

# QUANTUM GROUP-TWISTED TENSOR PRODUCTS OF C\*-ALGEBRAS

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**ABSTRACT.** We put two C\*-algebras together in a noncommutative tensor product using quantum group coactions on them and a bicharacter relating the two quantum groups that act. We describe this twisted tensor product in two equivalent ways. The first construction is based on certain pairs of representations of quantum groups which we call Heisenberg pairs because they generalise the Weyl form of the canonical commutation relations. The second construction uses covariant Hilbert space representations. We establish basic properties of the twisted tensor product and study some examples.

## 1. INTRODUCTION

Several important constructions put together two C\*-algebras in a kind of tensor product where the tensor factors do not commute. For instance, a noncommutative two-torus is obtained in this way from two copies of  $C(\mathbb{T})$ . More generally, the crossed product  $A \rtimes_{\alpha} G$  for a continuous action  $\alpha: G \rightarrow \text{Aut}(A)$  of a locally compact group  $G$  combines  $A$  and the group C\*-algebra of  $G$ . Such crossed products also exist for locally compact quantum groups. Another example is the graded tensor product for  $\mathbb{Z}/2$ -graded C\*-algebras, which is defined so that the odd elements anticommute. Ruy Exel began a more systematic study of such noncommutative tensor products in [3].

We shall construct twisted tensor products using quantum group coactions on the tensor factors. The examples mentioned above are special cases of our theory. Our construction is closely related to one by Nest and Voigt [10], but more general because we allow two different quantum groups to act on the tensor factors and do not need Haar weights on quantum groups.

Our twisted tensor product uses the following data: two C\*-quantum groups  $\mathbb{G} = (A, \Delta_A)$  and  $\mathbb{H} = (B, \Delta_B)$  (in the sense of [12]); a bicharacter  $V \in \mathcal{U}(\hat{A} \otimes \hat{B})$ ; and two C\*-algebras  $C$  and  $D$  with continuous coactions  $\gamma: C \rightarrow C \otimes A$  and  $\delta: D \rightarrow D \otimes B$  of  $\mathbb{G}$  and  $\mathbb{H}$ , respectively. Then we define a C\*-algebra

$$C \boxtimes_V D = (C, \gamma) \boxtimes_V (D, \delta)$$

with nondegenerate \*-homomorphisms

$$C \xrightarrow{\iota_C} \mathcal{M}(C \boxtimes_V D) \xleftarrow{\iota_D} D$$

such that  $\iota_C(C) \cdot \iota_D(D)$  is linearly dense in  $C \boxtimes_V D$ . We briefly call  $(C \boxtimes_V D, \iota_C, \iota_D)$  a crossed product of  $C$  and  $D$ .

We now give several examples.

First the trivial, commutative case. If  $V = 1$  or if  $\gamma$  or  $\delta$  is trivial, then  $C \boxtimes_V D$  is the minimal C\*-tensor product with the usual maps  $\iota_C$  and  $\iota_D$ .

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Secondly, let  $A = B = C^*(\mathbb{Z}/2)$ . Then  $C$  and  $D$  are  $\mathbb{Z}/2$ -graded  $C^*$ -algebras. Let  $V$  be the unique non-trivial bicharacter in  $\hat{A} \otimes \hat{B} = C(\mathbb{Z}/2 \times \mathbb{Z}/2)$ , defined by  $V(a, b) := a \cdot b$  for  $a, b \in \mathbb{Z}/2 = \{\pm 1\}$ . Then  $C \boxtimes_V D$  is the (spatial)  $\mathbb{Z}/2$ -graded tensor product: odd elements in  $C$  and  $D$  anticommute, and even elements commute with the other tensor factor.

Thirdly, let  $\mathbb{H} = \hat{\mathbb{G}}$  be the reduced dual of  $\mathbb{G}$  and let  $V = W^A \in \mathcal{U}(\hat{A} \otimes A)$  be the reduced bicharacter; here we identify the bidual of  $\mathbb{G}$  with  $\mathbb{G}$ . If  $\mathbb{G}$  has a Haar weight then our construction is equivalent to one by Nest and Voigt [10]. In particular, for  $D = \hat{A}$  and  $\delta = \hat{\Delta}_A$ ,  $C \boxtimes_{W^A} \hat{A}$  is the reduced crossed product for the coaction  $\gamma$ .

Finally, let  $A = B = C(\mathbb{T}^n)$ , so that coactions of  $\mathbb{G} = \mathbb{H}$  are actions of the  $n$ -torus group  $\mathbb{T}^n$ , and let  $C = D = C(\mathbb{T}^n)$  with  $\gamma = \delta = \Delta_A$ , corresponding to the translation action of  $\mathbb{T}^n$  on itself. A bicharacter  $V \in \mathcal{U}(\hat{A} \otimes \hat{B})$  is equivalent to a map  $\chi: \mathbb{Z}^n \times \mathbb{Z}^n \rightarrow \mathbb{T}$  that is multiplicative in both variables. Thus  $\chi((a_n), (b_n)) = \prod_{i,j=1}^n \lambda_{ij}^{a_i \cdot b_j}$  for some  $(\lambda_{ij})_{1 \leq i,j \leq n} \in \mathbb{T}$ . The resulting tensor product  $C(\mathbb{T}^n) \boxtimes_V C(\mathbb{T}^n)$  is generated by  $2n$  unitaries  $U_1, \dots, U_n$  and  $V_1, \dots, V_n$ , with the following commutation relations. First, the  $U_i$  and the  $V_i$  commute among themselves, so that they generate two copies of  $C(\mathbb{T}^n)$ . Secondly,  $V_i U_j = \lambda_{ij} U_j V_i$  for all  $1 \leq i, j \leq n$ . Thus we get all noncommutative  $2n$ -tori in this way.

Thus our construction unifies many different noncommutative tensor products. We will consider new examples of twisted tensor products in the sequel article [9], which is concerned with the case where  $\mathbb{G} = \mathbb{H}$  is a quasitriangular quantum group and  $V$  is its  $R$ -matrix. In this case,  $C \boxtimes_V D$  again carries a canonical coaction of  $\mathbb{G}$ , so that we may form tensor products of more than two factors. This tensor product turns out to be associative, so that we may ask for braided quantum groups with a comultiplication  $C \rightarrow C \boxtimes_V C$ . Such braided quantum groups may then be used to quantise semidirect product groups.

Now we describe our two constructions of  $C \boxtimes_V D$ .

The first one uses a pair of representations  $\alpha: A \rightarrow \mathbb{B}(\mathcal{H})$ ,  $\beta: B \rightarrow \mathbb{B}(\mathcal{H})$  on the same Hilbert space. This yields embeddings  $\iota_C := ((\text{id}_C \otimes \alpha) \circ \gamma)_{13}: C \rightarrow \mathcal{M}(C \otimes D \otimes \mathbb{K}(\mathcal{H}))$  and  $\iota_D := ((\text{id} \otimes \beta) \circ \delta)_{23}: D \rightarrow \mathcal{M}(C \otimes D \otimes \mathbb{K}(\mathcal{H}))$ . We let  $C \boxtimes_V D$  be the closed linear span of  $\iota_C(C) \cdot \iota_D(D)$  for a suitable choice of  $\alpha$  and  $\beta$ . We call suitable pairs *V-Heisenberg pairs*. The closed linear span of  $\iota_C(C) \cdot \iota_D(D)$  is a  $C^*$ -algebra, and different V-Heisenberg pairs  $(\alpha, \beta)$  yield equivalent crossed products.

The definition of a V-Heisenberg pair is a generalisation of the Weyl form of the canonical commutation relations (see Example 3.4). It is also a variant of the usual pentagon equation for multiplicative unitaries (see Example 3.9). In terms of the reduced bicharacters  $W^A \in \mathcal{U}(\hat{A} \otimes A)$  and  $W^B \in \mathcal{U}(\hat{B} \otimes B)$ , the pair  $(\alpha, \beta)$  is a V-Heisenberg pair if  $W_{1\alpha}^A W_{2\beta}^B = W_{2\beta}^B W_{1\alpha}^A V_{12}$  in  $\mathcal{U}(\hat{A} \otimes \hat{B} \otimes \mathbb{K}(\mathcal{H}))$ ; here  $W_{1\alpha}^A$  means that we apply  $\text{id} \otimes \text{id} \otimes \alpha$  to  $W_{13}^A$ .

Our second approach uses covariant representations of  $(C, \gamma)$  and  $(D, \delta)$  on Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$ . This contains corepresentations of  $A$  and  $B$ , which allow us to turn  $V$  into a unitary operator  $Z$  on  $\mathcal{H} \otimes \mathcal{K}$ . Assuming that the representations of  $C$  and  $D$  are faithful, we show that we get a faithful representation  $C \boxtimes_V D \rightarrow \mathbb{B}(\mathcal{H} \otimes \mathcal{K})$ , mapping  $C \ni c \mapsto c \otimes 1$  and  $D \ni d \mapsto Z(1 \otimes d)Z^*$ .

We also establish functoriality properties of  $\boxtimes$ . These say that a pair of equivariant “maps”  $f: C \rightarrow C'$ ,  $g: D \rightarrow D'$  induces a “map”  $C \boxtimes_V D \rightarrow C' \boxtimes_V D'$ . “Maps” could mean, among others, morphisms,  $*$ -homomorphisms, completely positive contractions, or  $C^*$ -correspondences. We also examine when  $f \boxtimes_V g$  is injective or

surjective. Functoriality for C\*-correspondences also shows that  $C \boxtimes_V D$  is Morita–Rieffel equivalent to  $C' \boxtimes_V D'$  if  $C, C'$  and  $D, D'$  are equivariantly Morita–Rieffel equivalent.

Related to Morita–Rieffel equivalence, we show that

$$(C, \gamma) \boxtimes_V (D, \delta) \cong (C, \gamma') \boxtimes_V (D, \delta')$$

if  $\gamma' = \text{Ad}_{u_\gamma} \circ \gamma$  and  $\delta' = \text{Ad}_{u_\delta} \circ \delta$  for cocycles  $u_\gamma \in \mathcal{U}(C \otimes A)$  and  $u_\delta \in \mathcal{U}(D \otimes B)$ . This generalises the well-known fact that the reduced crossed product for an inner group action and for the trivial action are isomorphic.

Finally, we consider the examples mentioned above. In particular, we show that  $(C, \gamma) \boxtimes_{W^A} (\hat{A}, \hat{\Delta}_A)$  is the reduced crossed product for the coaction  $\gamma$  on  $C$ . As an application of the functoriality of  $\boxtimes$ , we show how to construct the dual coaction on the reduced crossed product. The coaction  $\hat{\Delta}_A: \hat{A} \rightarrow \hat{A} \otimes \hat{A}$  is a  $\hat{\mathbb{G}}$ -equivariant map if we equip the codomain with the right  $\hat{\mathbb{G}}$ -coaction  $\text{id}_{\hat{A}} \otimes \hat{\Delta}_A$ . By functoriality of  $\boxtimes$ , it induces a morphism

$$(C, \gamma) \boxtimes_{W^A} (\hat{A}, \hat{\Delta}_A) \rightarrow (C, \gamma) \boxtimes_{W^A} (\hat{A} \otimes \hat{A}, \text{id}_{\hat{A}} \otimes \hat{\Delta}_A) \cong \hat{A} \otimes ((C, \gamma) \boxtimes_{W^A} (\hat{A}, \hat{\Delta}_A)).$$

It is easy to see that this is a continuous left  $\hat{A}$ -coaction on  $(C, \gamma) \boxtimes_{W^A} (\hat{A}, \hat{\Delta}_A)$ . This is exactly the dual coaction on the reduced crossed product.

## 2. PRELIMINARIES

For two norm-closed subsets  $X$  and  $Y$  of a C\*-algebra, let

$$X \cdot Y := \{xy : x \in X, y \in Y\}^{\text{CLS}},$$

where CLS stands for the *closed linear span*.

For a C\*-algebra  $A$ , let  $\mathcal{M}(A)$  be its multiplier algebra and let  $\mathcal{U}(A)$  be the group of unitary multipliers of  $A$ . A unitary  $U \in \mathcal{U}(A)$  defines an automorphism  $\text{Ad}_U \in \text{Aut}(A)$  by  $\text{Ad}_U(a) := UaU^*$ .

Let  $\mathfrak{C}^*\mathbf{alg}$  be the category of C\*-algebras with nondegenerate \*-homomorphisms  $\varphi: A \rightarrow \mathcal{M}(B)$  as morphisms  $A \rightarrow B$ ; let  $\text{Mor}(A, B)$  denote this set of morphisms.

Let  $\mathcal{H}$  be a Hilbert space. A *representation* of a C\*-algebra  $A$  is a nondegenerate \*-homomorphism  $A \rightarrow \mathbb{B}(\mathcal{H})$ . Since  $\mathbb{B}(\mathcal{H}) = \mathcal{M}(\mathbb{K}(\mathcal{H}))$  and the nondegeneracy conditions  $A \cdot \mathbb{K}(\mathcal{H}) = \mathbb{K}(\mathcal{H})$  and  $A \cdot \mathcal{H} = \mathcal{H}$  are equivalent, this is the same as a morphism from  $A$  to  $\mathbb{K}(\mathcal{H})$ .

### 2.1. Crossed tensor products.

**Definition 2.1** (compare [13]). Let  $A, B, C$  be C\*-algebras,  $\alpha \in \text{Mor}(A, C)$  and  $\beta \in \text{Mor}(B, C)$ . If  $\alpha(A) \cdot \beta(B) = C$ , then we call  $(C, \alpha, \beta)$  a *crossed product* or *crossed tensor product* of  $A$  and  $B$ .

*Example 2.2.* The spatial tensor product  $C = A \otimes B$  of two C\*-algebras with  $\alpha(a) = a \otimes 1_B$  and  $\beta(b) = 1_A \otimes b$  is the simplest example of a crossed product.

Let  $\alpha$  and  $\beta$  be (nondegenerate) representations of  $A$  and  $B$  on the same Hilbert space  $\mathcal{H}$  such that  $\alpha(A) \cdot \beta(B)$  and  $\beta(B) \cdot \alpha(A)$  are the same subspace of  $\mathbb{B}(\mathcal{H})$ . Then  $C := \alpha(A) \cdot \beta(B)$  is a C\*-algebra,  $\alpha \in \text{Mor}(A, C)$  and  $\beta \in \text{Mor}(B, C)$ . Thus  $C$  is a crossed product of  $A$  and  $B$ . This suggests that crossed products are defined by some commutation relations between  $\alpha$  and  $\beta$ . In a purely algebraic setting, Exel [3] describes the multiplication on  $C$  through the map  $\tau: B \otimes A \rightarrow A \otimes B$  expressing  $\beta(b)\alpha(a)$  as a linear combination of elements of the form  $\alpha(a')\beta(b')$ ; but this does not work for C\*-algebras because of the completions involved.

**Definition 2.3.** Two crossed products  $C_1 = \alpha_1(A) \cdot \beta_1(B)$  and  $C_2 = \alpha_2(A) \cdot \beta_2(B)$  of  $A$  and  $B$  are *equivalent* if there is a faithful morphism  $\varphi \in \text{Mor}(C_1, C_2)$  with  $\varphi \circ \alpha_1 = \alpha_2$  and  $\varphi \circ \beta_1 = \beta_2$ .

Any faithful morphism  $\varphi \in \text{Mor}(C_1, C_2)$  with  $\varphi \circ \alpha_1 = \alpha_2$  and  $\varphi \circ \beta_1 = \beta_2$  satisfies  $\varphi(C_1) = C_2$  and hence is a  $*$ -isomorphism.

*Example 2.4.* Let  $C = \alpha(A) \cdot \beta(B)$  be a crossed product and  $U \in \mathcal{U}(C)$ . Then

$$(C, \alpha, \beta) \simeq (C, \text{Ad}_U \circ \alpha, \text{Ad}_U \circ \beta).$$

## 2.2. Multiplicative unitaries and quantum groups.

**Definition 2.5** ([2]). Let  $\mathcal{H}$  be a Hilbert space. A unitary  $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$  is *multiplicative* if it satisfies the *pentagon equation*

$$(2.6) \quad \mathbb{W}_{23} \mathbb{W}_{12} = \mathbb{W}_{12} \mathbb{W}_{13} \mathbb{W}_{23} \quad \text{in } \mathcal{U}(\mathcal{H} \otimes \mathcal{H} \otimes \mathcal{H}).$$

Technical assumptions such as manageability ([14]) or, more generally, modularity ([11]) are needed in order to construct  $C^*$ -algebras out of a multiplicative unitary.

**Theorem 2.7** ([11, 12, 14]). *Let  $\mathcal{H}$  be a separable Hilbert space and  $\mathbb{W} \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$  a modular or manageable multiplicative unitary. Let*

$$(2.8) \quad A := \{(\omega \otimes \text{id}_{\mathcal{H}}) \mathbb{W} : \omega \in \mathbb{B}(\mathcal{H})_*\}^{\text{CLS}},$$

$$(2.9) \quad \hat{A} := \{(\text{id}_{\mathcal{H}} \otimes \omega) \mathbb{W} : \omega \in \mathbb{B}(\mathcal{H})_*\}^{\text{CLS}}.$$

- (1)  $A$  and  $\hat{A}$  are nondegenerate  $C^*$ -subalgebras of  $\mathbb{B}(\mathcal{H})$ .
- (2)  $\mathbb{W} \in \mathcal{U}(\hat{A} \otimes A) \subseteq \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$ . We write  $W^A$  for  $\mathbb{W}$  viewed as a unitary multiplier of  $\hat{A} \otimes A$  and call it reduced bicharacter.
- (3) There is a unique  $\Delta_A \in \text{Mor}(A, A \otimes A)$  such that

$$(2.10) \quad (\text{id}_{\hat{A}} \otimes \Delta_A) W^A = W_{12}^A W_{13}^A \quad \text{in } \mathcal{U}(\hat{A} \otimes A \otimes A);$$

it is coassociative:

$$(2.11) \quad (\Delta_A \otimes \text{id}_A) \circ \Delta_A = (\text{id}_A \otimes \Delta_A) \circ \Delta_A,$$

and satisfies the Podleś condition

$$(2.12) \quad \Delta_A(A) \cdot (1_A \otimes A) = A \otimes A = (A \otimes 1_A) \cdot \Delta_A(A).$$

- (4) There is a unique normal, linear, involutive anti-automorphism  $R_A$  of  $A$  such that

$$(2.13) \quad \Delta_A \circ R_A = \sigma \circ (R_A \otimes R_A) \circ \Delta_A.$$

where  $\sigma(x \otimes y) = y \otimes x$ .

A  $C^*$ -quantum group is a  $C^*$ -bialgebra  $\mathbb{G} = (A, \Delta_A)$  constructed from a modular or manageable multiplicative unitary. We do not need Haar weights.

The *dual* multiplicative unitary  $\widehat{\mathbb{W}} := \Sigma(\widehat{\mathbb{W}})^* \Sigma \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$  is modular or manageable if  $\mathbb{W}$  is, where  $\Sigma(x \otimes y) = y \otimes x$ . The  $C^*$ -quantum group  $\widehat{\mathbb{G}} = (\hat{A}, \hat{\Delta}_A)$  associated to  $\widehat{\mathbb{W}}$  is the *dual* of the quantum group  $\mathbb{G}$ . Its comultiplication is given by

$$(2.14) \quad (\hat{\Delta}_A \otimes \text{id}_A) W^A = W_{23}^A W_{13}^A \quad \text{in } \mathcal{U}(\hat{A} \otimes \hat{A} \otimes A).$$

Let  $\mathbb{G} = (A, \Delta_A)$  be a  $C^*$ -quantum.

**Definition 2.15.** A *continuous (right) coaction* of  $\mathbb{G}$  on a  $C^*$ -algebra  $C$  is a morphism  $\gamma: C \rightarrow C \otimes A$  with the following properties:

- (1)  $\gamma$  is injective;

(2)  $\gamma$  is a comodule structure, that is, the following diagram commutes:

$$(2.16) \quad \begin{array}{ccc} C & \xrightarrow{\gamma} & C \otimes A \\ \gamma \downarrow & & \downarrow \text{id}_C \otimes \Delta_A \\ C \otimes A & \xrightarrow{\gamma \otimes \text{id}_A} & C \otimes A \otimes A \end{array}$$

(3)  $\gamma$  satisfies the *Podleś condition*  $\gamma(C) \cdot (1_C \otimes A) = C \otimes A$ .

We call  $(C, \gamma)$  a  $\mathbb{G}$ -C\*-algebra. We often drop  $\gamma$  from our notation.

A morphism  $f: C \rightarrow D$  between two  $\mathbb{G}$ -C\*-algebras  $(C, \gamma)$  and  $(D, \delta)$  is  $\mathbb{G}$ -equivariant if  $\delta \circ f = (f \otimes \text{id}_A) \circ \gamma$ . Let  $\text{Mor}^{\mathbb{G}}(C, D)$  be the set of  $\mathbb{G}$ -equivariant morphisms from  $C$  to  $D$ . Let  $\mathfrak{C}^*\text{alg}(\mathbb{G})$  be the category with  $\mathbb{G}$ -C\*-algebras as objects and  $\mathbb{G}$ -equivariant morphisms as arrows.

*Example 2.17.* The *trivial*  $\mathbb{G}$ -coaction on a C\*-algebra  $C$  is  $\tau: C \rightarrow C \otimes A$ ,  $c \mapsto c \otimes 1_A$ . It is always continuous. Theorem 2.7.3 implies that  $\Delta_A: A \rightarrow A \otimes A$  is a continuous  $\mathbb{G}$ -coaction on  $A$  for any C\*-quantum group  $\mathbb{G} = (A, \Delta_A)$ . More generally,  $\text{id}_C \otimes \Delta_A: C \otimes A \rightarrow C \otimes A \otimes A$  is a continuous  $\mathbb{G}$ -coaction on  $C \otimes A$  for any C\*-algebra  $C$ . The following lemma says that any continuous coaction may be embedded into one of this form.

**Lemma 2.18.** *Let  $C$  be a C\*-algebra and  $D$  a C\*-subalgebra of  $\mathcal{M}(C \otimes A)$  with*

$$(2.19) \quad (\text{id}_C \otimes \Delta_A)(D) \cdot (1_{C \otimes A} \otimes A) = D \otimes A.$$

*Then  $D$  with the coaction  $\delta := (\text{id}_C \otimes \Delta_A)|_D: D \rightarrow D \otimes A$  is a  $\mathbb{G}$ -C\*-algebra, and the embedding  $D \rightarrow \mathcal{M}(C \otimes A)$  is a  $\mathbb{G}$ -equivariant morphism.*

*Every  $\mathbb{G}$ -C\*-algebra is isomorphic to one of this form.*

*Proof.* Equation (2.19) implies that  $\text{id}_C \otimes \Delta_A$  maps  $D$  into  $\mathcal{M}(D \otimes A)$  as claimed. Injectivity and coassociativity of  $\delta$  are clear because they hold for  $\text{id}_C \otimes \Delta_A$ , and (2.19) is the Podleś condition for  $\delta$ . Thus  $\delta$  is a continuous  $\mathbb{G}$ -coaction. The equivariance of the embedding is clear.

Now let  $(C, \gamma)$  be a  $\mathbb{G}$ -C\*-algebra. Let  $D := \gamma(C) \subseteq \mathcal{M}(C \otimes A)$ . The comodule property (2.16) and the Podleś condition for  $\gamma$  imply that  $D$  satisfies (2.19):

$$\begin{aligned} (\text{id}_C \otimes \Delta_A)\gamma(C) \cdot (1_{C \otimes A} \otimes A) &= (\gamma \otimes \text{id}_A)(\gamma(C) \cdot (1_C \otimes A)) \\ &= (\gamma \otimes \text{id}_A)(C \otimes A) = \gamma(C) \otimes A. \end{aligned}$$

Furthermore, the comodule property (2.16) for  $\gamma$  implies that the isomorphism  $\gamma: C \rightarrow D$  is  $\mathbb{G}$ -equivariant.  $\square$

**Definition 2.20.** A (right) *corepresentation* of  $\mathbb{G}$  on a Hilbert space  $\mathcal{H}$  is a unitary  $U \in \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes A)$  with

$$(2.21) \quad (\text{id}_D \otimes \Delta_A)U = U_{12}U_{13} \quad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes A \otimes A).$$

**Definition 2.22.** A *covariant representation* of  $(C, \gamma, A)$  on a Hilbert space  $\mathcal{H}$  is a pair consisting of a corepresentation  $U \in \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes A)$  and a representation  $\varphi: C \rightarrow \mathbb{B}(\mathcal{H})$  that satisfy the covariance condition

$$(2.23) \quad (\varphi \otimes \text{id}_A) \circ \gamma(c) = U(\varphi(c) \otimes 1_A)U^* \quad \text{in } \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes A)$$

for all  $c \in C$ . A *faithful covariant representation* is one where  $\varphi$  is faithful.

**2.3. Bicharacters as quantum group morphisms.** Let  $\mathbb{G} = (A, \Delta_A)$  and  $\mathbb{H} = (B, \Delta_B)$  be  $C^*$ -quantum groups, let  $\hat{\mathbb{G}} = (\hat{A}, \hat{\Delta}_A)$  and  $\hat{\mathbb{H}} = (\hat{B}, \hat{\Delta}_B)$  be their duals.

**Definition 2.24** ([8, Definition 16]). A *bicharacter* from  $\mathbb{G}$  to  $\hat{\mathbb{H}}$  is a unitary  $V \in \mathcal{U}(\hat{A} \otimes \hat{B})$  with

$$(2.25) \quad (\hat{\Delta}_A \otimes \text{id}_{\hat{B}})V = V_{23}V_{13} \quad \text{in } \mathcal{U}(\hat{A} \otimes \hat{A} \otimes \hat{B}),$$

$$(2.26) \quad (\text{id}_{\hat{A}} \otimes \hat{\Delta}_B)V = V_{12}V_{13} \quad \text{in } \mathcal{U}(\hat{A} \otimes \hat{B} \otimes \hat{B}).$$

It is shown in [8] that bicharacters in  $\mathcal{U}(\hat{A} \otimes B)$  may be viewed as morphisms of quantum groups from  $A$  to  $B$ . We shall use bicharacters in  $\mathcal{U}(\hat{A} \otimes \hat{B})$  throughout and rewrite some definitions in [8] in this setting.

**Definition 2.27.** A *right quantum group morphism* from  $\mathbb{G}$  to  $\hat{\mathbb{H}}$  is a morphism  $\Delta_R: A \rightarrow A \otimes \hat{B}$  such that the following diagrams commute:

$$(2.28) \quad \begin{array}{ccc} A & \xrightarrow{\Delta_R} & A \otimes \hat{B} \\ \Delta_A \downarrow & & \downarrow \Delta_A \otimes \text{id}_{\hat{B}} \\ A \otimes A & \xrightarrow{\text{id}_A \otimes \Delta_R} & A \otimes A \otimes \hat{B} \end{array} \quad \begin{array}{ccc} A & \xrightarrow{\Delta_R} & A \otimes \hat{B} \\ \Delta_R \downarrow & & \downarrow \text{id}_A \otimes \hat{\Delta}_B \\ A \otimes \hat{B} & \xrightarrow{\Delta_R \otimes \text{id}_{\hat{B}}} & A \otimes \hat{B} \otimes \hat{B} \end{array}$$

The following theorem summarises some of the main results of [8].

**Theorem 2.29.** *There are natural bijections between the following sets:*

- (1) *bicharacters  $V \in \mathcal{U}(\hat{A} \otimes \hat{B})$  from  $\mathbb{G}$  to  $\hat{\mathbb{H}}$ ;*
- (2) *bicharacters  $\hat{V} := \sigma(V^*) \in \mathcal{U}(\hat{B} \otimes \hat{A})$  from  $\mathbb{H}$  to  $\hat{\mathbb{G}}$ ;*
- (3) *right quantum group homomorphisms  $\Delta_R: A \rightarrow A \otimes \hat{B}$ ;*
- (4) *functors  $F: \mathfrak{C}^*\text{alg}(\mathbb{G}) \rightarrow \mathfrak{C}^*\text{alg}(\hat{\mathbb{H}})$  with  $\text{For}_{\hat{\mathbb{H}}} \circ F = \text{For}_{\mathbb{G}}$  for the forgetful functor  $\text{For}_{\mathbb{G}}: \mathfrak{C}^*\text{alg}(\mathbb{G}) \rightarrow \mathfrak{C}^*\text{alg}$ ;*
- (5) *Hopf morphisms  $f: A^u \rightarrow \hat{B}^u$  between universal quantum groups.*

The first bijection maps a bicharacter  $V$  to

$$(2.30) \quad \hat{V} := \sigma(V^*).$$

A bicharacter  $V$  and a right quantum group homomorphism  $\Delta_R$  determine each other uniquely via

$$(2.31) \quad (\text{id}_{\hat{A}} \otimes \Delta_R)(W^A) = W_{12}^A V_{13}.$$

The functor  $F$  associated to  $\Delta_R$  is the unique one that maps  $(A, \Delta_A)$  to  $(A, \Delta_R)$ . In general,  $F$  maps a continuous  $\mathbb{G}$ -coaction  $\gamma: C \rightarrow C \otimes A$  to the unique  $\hat{\mathbb{H}}$ -coaction  $\delta: C \rightarrow C \otimes \hat{B}$  for which the following diagram commutes:

$$(2.32) \quad \begin{array}{ccc} C & \xrightarrow{\gamma} & C \otimes A \\ \delta \downarrow & & \downarrow \text{id}_C \otimes \Delta_R \\ C \otimes \hat{B} & \xrightarrow{\gamma \otimes \text{id}_{\hat{B}}} & C \otimes A \otimes \hat{B} \end{array}$$

The bicharacter in  $\mathcal{U}(\hat{A} \otimes \hat{B})$  associated to a Hopf morphism  $f: A^u \rightarrow \hat{B}^u$  is  $V := (\text{id}_{\hat{A}} \otimes \Lambda_{\hat{B}} f)(W^A)$ , where  $W^A \in \mathcal{U}(\hat{A} \otimes A^u)$  is the unique bicharacter lifting  $W^A \in \mathcal{U}(\hat{A} \otimes A)$  and  $\Lambda_{\hat{B}}: \hat{B}^u \rightarrow \hat{B}$  is the reducing map.

## 3. HEISENBERG PAIRS AND TWISTED TENSOR PRODUCTS

This section introduces Heisenberg and anti-Heisenberg pairs, and uses them to construct our noncommutative tensor product, after establishing properties of Heisenberg pairs necessary for that purpose.

Let  $\mathbb{G} = (A, \Delta_A)$  and  $\mathbb{H} = (B, \Delta_B)$  be  $C^*$ -quantum groups. Let  $W^A \in \mathcal{U}(\hat{A} \otimes A)$  and  $W^B \in \mathcal{U}(\hat{B} \otimes B)$  be their reduced bicharacters. Let  $V \in \mathcal{U}(\hat{A} \otimes \hat{B})$  be a bicharacter from  $A$  to  $\hat{B}$ . Heisenberg pairs and anti-Heisenberg pairs are pairs of representations  $(\alpha, \beta)$  of  $\hat{A}$  and  $\hat{B}$  on the same Hilbert space  $\mathcal{H}$  that satisfy suitable compatibility conditions.

We use these pairs to define twisted tensor products  $C \boxtimes_V D$  in Section 3.2. A crucial technical point is to show a pair of representations of  $C$  and  $D$  generates a crossed product  $C^*$ -algebra. Here the commutativity result in Section 3.1 is crucial. In addition, we construct examples of V-Heisenberg pairs and V-anti-Heisenberg, proving their existence, and give some equivalent characterisations of them.

**Definition 3.1.** A pair of representations  $\alpha: A \rightarrow \mathbb{B}(\mathcal{H})$ ,  $\beta: B \rightarrow \mathbb{B}(\mathcal{H})$  is called a *V-Heisenberg pair* or briefly *Heisenberg pair* if

$$(3.2) \quad W_{1\alpha}^A W_{2\beta}^B = W_{2\beta}^B W_{1\alpha}^A V_{12} \quad \text{in } \mathcal{U}(\hat{A} \otimes \hat{B} \otimes \mathbb{K}(\mathcal{H}));$$

here  $W_{1\alpha}^A := ((\text{id}_{\hat{A}} \otimes \alpha)W^A)_{13}$  and  $W_{2\beta}^B := ((\text{id}_{\hat{B}} \otimes \beta)W^B)_{23}$ . It is called a *V-anti-Heisenberg pair* or briefly *anti-Heisenberg pair* if

$$(3.3) \quad W_{2\beta}^B W_{1\alpha}^A = V_{12} W_{1\alpha}^A W_{2\beta}^B \quad \text{in } \mathcal{U}(\hat{A} \otimes \hat{B} \otimes \mathbb{K}(\mathcal{H})),$$

with similar conventions as above.

We name these pairs after Heisenberg because of the following example:

*Example 3.4.* Let  $A = B = C^*(\mathbb{R})$  be the group  $\mathbb{R}$  viewed as a quantum group, and let  $V \in \mathcal{U}(\hat{A} \otimes \hat{B}) \cong C(\mathbb{R} \times \mathbb{R}, \mathbb{T})$  be the standard bicharacter  $(s, t) \mapsto \exp(ist)$ . A pair of representations of  $A$  is a pair of unitary one-parameter groups  $(U_1(s), U_2(t))_{s,t \in \mathbb{R}}$ . Equation (3.2) is equivalent to the canonical commutation relation in the *Weyl form*:

$$U_2(t)U_1(s) = \exp(-ist)U_1(s)U_2(t) \quad \text{for all } s, t \in \mathbb{R}.$$

The case where  $\mathbb{H} = \hat{\mathbb{G}}$  and  $V = W^A \in \mathcal{U}(\hat{A} \otimes A)$  is the reduced bicharacter of  $\mathbb{G}$  is particularly interesting:

**Definition 3.5.** A  $W^A$ -Heisenberg or  $W^A$ -anti-Heisenberg pair is also called a  $\mathbb{G}$ -Heisenberg pair or  $\mathbb{G}$ -anti-Heisenberg pair, respectively.

**Lemma 3.6.** A pair of representations  $(\pi, \hat{\pi})$  of  $A$  and  $\hat{A}$  on  $\mathcal{H}$  is a  $\mathbb{G}$ -Heisenberg pair if and only if

$$(3.7) \quad W_{\hat{\pi}3}^A W_{1\pi}^A = W_{1\pi}^A W_{13}^A W_{\hat{\pi}3}^A \quad \text{in } \mathcal{U}(\hat{A} \otimes \mathbb{K}(\mathcal{H}) \otimes A).$$

It is a  $\mathbb{G}$ -anti-Heisenberg pair if and only if

$$(3.8) \quad W_{1\pi}^A W_{\hat{\pi}3}^A = W_{\hat{\pi}3}^A W_{13}^A W_{1\pi}^A \quad \text{in } \mathcal{U}(\hat{A} \otimes \mathbb{K}(\mathcal{H}) \otimes A).$$

*Proof.* Let  $\pi$  and  $\hat{\pi}$  be representations of  $A$  and  $\hat{A}$  on  $\mathcal{H}$  satisfying (3.7). When we apply  $\sigma_{23}$  to both sides of (3.7) we get

$$(\widehat{W}_{2\hat{\pi}}^A)^* W_{1\pi}^A = W_{1\pi}^A W_{12}^A (\widehat{W}_{2\hat{\pi}}^A)^* \quad \text{in } \mathcal{U}(\hat{A} \otimes A \otimes \mathbb{K}(\mathcal{H})).$$

This is equivalent to  $W_{1\pi}^A \widehat{W}_{2\hat{\pi}}^A = \widehat{W}_{2\hat{\pi}}^A W_{1\pi}^A W_{12}^A$ , which is (3.2) for  $\hat{B} = A$ ,  $V = W^A$ ,  $\alpha = \pi$  and  $\beta = \hat{\pi}$ . This computation may be reversed as well.

The computation for anti-Heisenberg pairs is similar.  $\square$

*Example 3.9.* If  $W \in \mathcal{U}(\mathcal{H} \otimes \mathcal{H})$  is a modular multiplicative unitary generating  $\mathbb{G} = (A, \Delta_A)$ , then the resulting representations  $\pi: A \rightarrow \mathbb{B}(\mathcal{H})$  and  $\hat{\pi}: \hat{A} \rightarrow \mathbb{B}(\mathcal{H})$  form a  $\mathbb{G}$ -Heisenberg pair: the condition (3.7) is equivalent to the pentagon equation (2.6) for  $W_{\hat{\pi}\pi}^A := (\hat{\pi} \otimes \pi)(W^A)$ . Conversely, a pair of faithful representations  $(\pi, \hat{\pi})$  is a  $\mathbb{G}$ -Heisenberg pair if and only if  $W_{\hat{\pi}\pi}^A$  is a multiplicative unitary.

Let  $\overline{\mathcal{H}}$  be the conjugate Hilbert space to the Hilbert space  $\mathcal{H}$ . Recall that the *transpose* of an operator  $x \in \mathbb{B}(\mathcal{H})$  is the operator  $x^\top \in \mathbb{B}(\overline{\mathcal{H}})$  defined by  $x^\top(\xi) := \overline{x^*\xi}$  for all  $\xi \in \mathcal{H}$ . The transposition is a linear, involutive anti-automorphism  $\mathbb{B}(\mathcal{H}) \rightarrow \mathbb{B}(\overline{\mathcal{H}})$ . Recall that the unitary antipode  $R_A: A \rightarrow A$  is also a linear, involutive anti-automorphism. Therefore, if  $\alpha: A \rightarrow \mathbb{B}(\mathcal{H})$  and  $\beta: B \rightarrow \mathbb{B}(\mathcal{H})$  are representations, then so are

$$\begin{aligned}\bar{\alpha}: A &\rightarrow \mathbb{B}(\overline{\mathcal{H}}), & a &\mapsto (R_A(a))^\top, \\ \bar{\beta}: B &\rightarrow \mathbb{B}(\overline{\mathcal{H}}), & b &\mapsto (R_B(b))^\top.\end{aligned}$$

**Lemma 3.10.** *The pair  $(\alpha, \beta)$  is Heisenberg if and only if  $(\bar{\alpha}, \bar{\beta})$  is anti-Heisenberg.*

*Proof.* Let  $(\alpha, \beta)$  be a Heisenberg pair. The following computation shows that  $(\bar{\alpha}, \bar{\beta})$  is an anti-Heisenberg pair:

$$\begin{aligned}W_{2\bar{\beta}}^B W_{1\bar{\alpha}}^A &= (R_{\hat{A}} \otimes R_{\hat{B}} \otimes T)(W_{1\alpha}^A W_{2\beta}^B) \\ &= (R_{\hat{A}} \otimes R_{\hat{B}} \otimes T)(W_{2\beta}^B W_{1\alpha}^A V_{12}) = V_{12} W_{1\alpha}^A W_{2\beta}^B;\end{aligned}$$

the first equality uses  $(R_{\hat{A}} \otimes R_{\hat{B}})V = V$  ([8, Proposition 3.10]) and the antimultiplicativity of  $R_{\hat{A}} \otimes R_{\hat{B}} \otimes T$ ; the second equality uses (3.2); and the third equality uses the same facts as for the previous ones in reverse order.

The computation above may be reversed to show the converse direction.  $\square$

Thus Heisenberg pairs and anti-Heisenberg pairs are essentially equivalent.

Recall that a bicharacter  $V$  gives rise to a dual bicharacter  $\hat{V} \in \mathcal{U}(\hat{B} \otimes \hat{A})$  by (2.30) and to a right quantum group homomorphism  $\Delta_R: A \rightarrow A \otimes \hat{B}$  with (2.31). Similarly,  $\hat{V}$  gives rise to a right quantum group homomorphism  $\hat{\Delta}_R: B \rightarrow B \otimes \hat{A}$ . We may reformulate the condition of being a Heisenberg pair in terms of  $\hat{V}$ ,  $\Delta_R$  and  $\hat{\Delta}_R$ , respectively:

**Lemma 3.11.** *Let  $\alpha$  and  $\beta$  be representations of  $A$  and  $B$  on a Hilbert space  $\mathcal{H}$ . Then the following are equivalent:*

- (1)  $(\alpha, \beta)$  is a Heisenberg pair acting on  $\mathcal{H}$ ;
- (2)  $(\beta, \alpha)$  is a  $\hat{V}$ -Heisenberg pair acting on  $\mathcal{H}$ ;
- (3)  $(\alpha \otimes \text{id}_{\hat{B}})\Delta_R(a) = (\widehat{W}_{\beta 2}^B)(\alpha(a) \otimes 1_{\hat{B}})(\widehat{W}_{\beta 2}^B)^*$  for all  $a \in A$ ;
- (4)  $(\beta \otimes \text{id}_{\hat{A}})\hat{\Delta}_R(b) = (\widehat{W}_{\alpha 2}^A)(\beta(b) \otimes 1_{\hat{A}})(\widehat{W}_{\alpha 2}^A)^*$  for all  $b \in B$ .

*Proof.* (1)  $\iff$  (2): (1) is equivalent to

$$W_{1\alpha}^A W_{2\beta}^B V_{12}^* = W_{2\beta}^B W_{1\alpha}^A \quad \text{in } \mathcal{U}(\hat{A} \otimes \hat{B} \otimes \mathbb{K}(\mathcal{H}))$$

by (3.2). Applying  $\sigma_{12}$  gives

$$(3.12) \quad W_{2\alpha}^A W_{1\beta}^B \hat{V}_{12} = W_{1\beta}^B W_{2\alpha}^A \quad \text{in } \mathcal{U}(\hat{B} \otimes \hat{A} \otimes \mathbb{K}(\mathcal{H})),$$

which is equivalent to  $(\beta, \alpha)$  being a  $\hat{V}$ -Heisenberg pair. Thus (1)  $\iff$  (2).

(1)  $\iff$  (3): Let  $(\alpha, \beta)$  be a Heisenberg pair. The following computation takes place in  $\mathcal{U}(\hat{A} \otimes \mathbb{K}(\mathcal{H}) \otimes \hat{B})$ :

$$\begin{aligned}(\text{id}_{\hat{A}} \otimes \alpha \otimes \text{id}_{\hat{B}})(\text{id}_{\hat{A}} \otimes \Delta_R)W^A &= W_{1\alpha}^A V_{13} = \sigma_{23}(W_{1\alpha}^A V_{12}) \\ &= \sigma_{23}((W_{2\beta}^B)^* W_{1\alpha}^B W_{2\beta}^B) = (\widehat{W}_{\beta 3}^B)W_{1\alpha}^A (\widehat{W}_{\beta 3}^B)^*;\end{aligned}$$



the first equality uses (2.31); the second equality is obvious; the third equality uses (3.2); and the last equality uses  $\widehat{W}^B = \sigma((W^B)^*)$ . Since  $\{(\omega \otimes \text{id}_A)W^A : \omega \in \hat{A}'\}$  is linearly dense in  $A$ , slicing the first leg of the first and the last expression of the above expression yields (1) $\implies$ (3).

Conversely, applying  $\text{id}_{\hat{A}} \otimes \alpha \otimes \text{id}_{\hat{B}}$  on both sides of (2.31) and using (3), we get

$$W_{1\alpha}^A V_{13} = (\text{id}_{\hat{A}} \otimes (\alpha \otimes \text{id}_{\hat{B}}) \Delta_R) W^A = (\widehat{W}_{\beta 3}^B) W_{1\alpha}^A (\widehat{W}_{\beta 3}^B)^* \quad \text{in } \mathcal{U}(\hat{A} \otimes \mathbb{K}(\mathcal{H}) \otimes \hat{B});$$

applying  $\sigma_{23}$  to this gives (3.2). Thus (3) $\implies$ (1).

To prove (2)  $\iff$  (4) argue as in the proof that (1)  $\iff$  (3).  $\square$

**Lemma 3.13.** *Let  $(\pi, \hat{\pi})$  and  $(\eta, \hat{\eta})$  be  $\mathbb{G}$ - and  $\mathbb{H}$ -Heisenberg pairs on Hilbert spaces  $\mathcal{H}_\pi$  and  $\mathcal{H}_\eta$ , respectively. Then the pair of representations  $(\alpha, \beta)$  of  $A$  and  $B$  on  $\mathcal{H}_\pi \otimes \mathcal{H}_\eta$  defined by  $\alpha(a) := (\pi \otimes \hat{\eta}) \Delta_R(a)$  and  $\beta(b) := 1_{\mathcal{H}_\pi} \otimes \eta(b)$  is a  $V$ -Heisenberg pair; here  $\Delta_R: A \rightarrow A \otimes \hat{B}$  is the right quantum group homomorphism associated to  $V$  as in Theorem 2.29.*

*Proof.* First we check the following equation:

$$(3.14) \quad V_{1\hat{\eta}} W_{2\eta}^B = W_{2\eta}^B V_{1\hat{\eta}} V_{12} \quad \text{in } \mathcal{U}(\hat{A} \otimes \hat{B} \otimes \mathbb{K}(\mathcal{H}_\eta)).$$

The coaction  $\hat{B} \rightarrow \hat{B} \otimes \hat{B}$  associated to the reduced bicharacter  $W^B$  is the usual comultiplication  $\hat{\Delta}_B$ . Hence

$$(\widehat{W}_{\eta 3}^B) V_{1\hat{\eta}} (\widehat{W}_{\eta 3}^B)^* = (\text{id}_{\hat{A}} \otimes \hat{\eta} \otimes \text{id}_{\hat{B}}) (\text{id} \otimes \hat{\Delta}_B) V = (\text{id}_{\hat{A}} \otimes \hat{\eta} \otimes \text{id}_{\hat{B}}) (V_{12} V_{13}) = V_{1\hat{\eta}} V_{13}$$

in  $\mathcal{U}(\hat{A} \otimes \mathbb{K}(\mathcal{H}_\eta) \otimes \hat{B})$  because of Lemma 3.11(4) and the bicharacter property (2.26) of  $V$ . When we flip the last two legs, we turn  $\widehat{W}_{\eta 3}^B$  into  $(W_{2\eta}^B)^*$ . Rearranging then gives (3.14).

Now we can check that  $(\alpha, \beta)$  is a Heisenberg pair. The following computation takes place in  $\mathcal{U}(\hat{A} \otimes \hat{B} \otimes \mathbb{K}(\mathcal{H}_\pi) \otimes \mathbb{K}(\mathcal{H}_\eta))$ :

$$W_{1\alpha}^A W_{2\beta}^B = W_{1\pi}^A V_{1\hat{\eta}} W_{2\eta}^B = W_{1\pi}^A W_{2\eta}^B V_{1\hat{\eta}} V_{12} = W_{2\eta}^B W_{1\pi}^A V_{1\hat{\eta}} V_{12} = W_{2\beta}^B W_{1\alpha}^A V_{12}$$

the first equality uses the definitions of  $\alpha$  and  $\beta$  and (2.31); the second equality uses (3.14); the third equality uses that  $W_{1\pi}^A$  and  $W_{2\eta}^B$  commute; and the fourth equality uses the definitions of  $\alpha$  and  $\beta$  again.  $\square$

**3.1. Commutativity and Heisenberg pairs.** Locality principles in quantum field theory always require commutation relations of the simplest possible form  $xy = yx$ . Our noncommutative tensor products are based on more complicated commutation relations. Nevertheless, we are going to show that ordinary commutativity comes out if we put together a Heisenberg and an anti-Heisenberg pair. This will be crucial for the existence of our noncommutative tensor product.

**Proposition 3.15.** *Let  $\mathcal{H}$  and  $\mathcal{K}$  be Hilbert spaces; let  $\alpha$  and  $\beta$  be representations of  $A$  and  $B$  on  $\mathcal{H}$ , respectively; and let  $\bar{\alpha}$  and  $\bar{\beta}$  be representations of  $A$  and  $B$  on  $\mathcal{K}$ , respectively. Then the following are equivalent:*

- (1) *the representations  $(\alpha \otimes \bar{\alpha}) \Delta_A$  and  $(\beta \otimes \bar{\beta}) \Delta_B$  of  $A$  and  $B$  on  $\mathcal{H} \otimes \mathcal{K}$  commute, that is, for any  $a \in A$  and  $b \in B$ , we have*

$$(3.16) \quad [(\alpha \otimes \bar{\alpha}) \Delta_A(a), (\beta \otimes \bar{\beta}) \Delta_B(b)] = 0;$$

- (2) *there is a bicharacter  $V \in \mathcal{U}(\hat{A} \otimes \hat{B})$  such that  $(\alpha, \beta)$  is a Heisenberg pair and  $(\bar{\alpha}, \bar{\beta})$  is an anti-Heisenberg pair.*

*Proof.* Equation (3.16) is equivalent to

$$(3.17) \quad W_{1\alpha}^A W_{1\bar{\alpha}}^A W_{2\beta}^B W_{2\bar{\beta}}^B = W_{2\beta}^B W_{2\bar{\beta}}^B W_{1\alpha}^A W_{1\bar{\alpha}}^A \quad \text{in } \mathcal{U}(\hat{A} \otimes \hat{B} \otimes \mathbb{K}(\mathcal{H}) \otimes \mathbb{K}(\mathcal{K}))$$

because of (2.10) and (2.8) for  $W^A$  and  $W^B$ . We may commute  $W_{1\bar{\alpha}}^A$  with  $W_{2\beta}^B$  and  $W_{2\bar{\beta}}^B$  with  $W_{1\alpha}^A$  on both sides and rearrange (3.17) to

$$(3.18) \quad V := (W^A)_{1\alpha}^* (W^B)_{2\beta}^* W_{1\alpha}^A W_{2\beta}^B = W_{2\bar{\beta}}^B W_{1\bar{\alpha}}^A (W^B)_{2\bar{\beta}}^* (W^A)_{1\bar{\alpha}}^*.$$

We have  $V \in \mathcal{U}(\hat{A} \otimes \hat{B})$  because its first definition has  $1_{\mathcal{K}}$  in the fourth leg and its second definition has  $1_{\mathcal{H}}$  in the third leg.

We check that  $V$  is a bicharacter in both legs. First we check (2.25):

$$\begin{aligned} (\hat{\Delta}_A \otimes \text{id}_{\hat{B}})V &= (\hat{\Delta}_A \otimes \text{id}_{\hat{B}})(W_{2\bar{\beta}}^B W_{1\bar{\alpha}}^A (W^B)_{2\bar{\beta}}^* (W^A)_{1\bar{\alpha}}^*) \\ &= W_{3\bar{\beta}}^B W_{2\bar{\alpha}}^A W_{1\bar{\alpha}}^A (W^B)_{3\bar{\beta}}^* (W^A)_{1\bar{\alpha}}^* (W^A)_{2\bar{\alpha}}^* \\ &= V_{23} W_{2\bar{\alpha}}^A V_{13} (W^A)_{2\bar{\alpha}}^* = V_{23} V_{13}; \end{aligned}$$

the first and third equality use the second definition of  $V$  in (3.18); the second equality uses (2.25) for  $W^A$ ; and the last equality uses that  $W_{2\bar{\alpha}}^A$  and  $V_{13}$  commute. A similar computation using the first definition of  $V$  in (3.18) yields  $(\text{id}_{\hat{A}} \otimes \hat{\Delta}_B)V = V_{12} V_{13}$ ; thus  $V \in \mathcal{U}(\hat{A} \otimes \hat{B})$  is a bicharacter.

The first definition of  $V$  in (3.18) says that  $(\alpha, \beta)$  is a Heisenberg pair, and the second one says that  $(\gamma, \delta)$  is an anti-Heisenberg pair.

Conversely, if  $V \in \mathcal{U}(\hat{A} \otimes \hat{B})$  is a bicharacter,  $(\alpha, \beta)$  a Heisenberg pair and  $(\gamma, \delta)$  an anti-Heisenberg pair, then (3.18) follows, and hence (3.16).  $\square$

**3.2. Twisted tensor products via Heisenberg pairs.** Let  $\mathbb{G} = (A, \Delta_A)$  and  $\mathbb{H} = (B, \Delta_B)$  be  $C^*$ -quantum groups, let  $V \in \mathcal{U}(\hat{A} \otimes \hat{B})$  be a bicharacter, let  $(C, \gamma)$  be a  $\mathbb{G}$ - $C^*$ -algebra, and let  $(D, \delta)$  be a  $\mathbb{H}$ - $C^*$ -algebra. Let  $(\alpha, \beta)$  be a  $V$ -Heisenberg pair on some Hilbert space  $\mathcal{H}$ .

Using this data, we now construct a crossed product  $(C \boxtimes_V D, \iota_C, \iota_D)$  of  $C$  and  $D$  in the sense of the Definition 2.1. A more precise notation is

$$C \boxtimes_V D = (C, \gamma) \boxtimes_V (D, \delta).$$

There is no need to mention  $(\alpha, \beta)$  in our notation because all Heisenberg pairs give equivalent crossed products; we will prove this in Section 4.

**Lemma 3.19.** *Define morphisms*

$$\begin{aligned} \iota_C: C &\rightarrow C \otimes D \otimes \mathbb{K}(\mathcal{H}), & c &\mapsto (\text{id}_C \otimes \alpha)\gamma(c)_{13}, \\ \iota_D: D &\rightarrow C \otimes D \otimes \mathbb{K}(\mathcal{H}), & d &\mapsto (\text{id}_D \otimes \beta)\delta(d)_{23}. \end{aligned}$$

Then  $\iota_C(C) \cdot \iota_D(D) = \iota_D(D) \cdot \iota_C(C)$  in  $\mathcal{M}(C \otimes D \otimes \mathbb{K}(\mathcal{H}))$ .

It follows that

$$C \boxtimes_V D := \iota_C(C) \cdot \iota_D(D)$$

is a  $C^*$ -algebra and that  $\iota_C$  and  $\iota_D$  are morphisms from  $C$  and  $D$  to  $C \boxtimes_V D$ , respectively. Thus  $(C \boxtimes_V D, \iota_C, \iota_D)$  is a crossed product of  $C$  and  $D$  in the sense of Definition 2.1.

The extension of  $\boxtimes_V$  to Hilbert modules uses the following generalisation:

**Lemma 3.20.** *Let  $X \subseteq C$  and  $Y \subseteq D$  be closed subspaces with*

$$\gamma(X) \cdot (1_C \otimes A) = X \otimes A \quad \text{and} \quad \delta(Y) \cdot (1_D \otimes B) = Y \otimes B.$$

Then  $\iota_C(X) \cdot \iota_Y(D) = \iota_D(Y) \cdot \iota_C(X)$  in  $\mathcal{M}(C \otimes D \otimes \mathbb{K}(\mathcal{H}))$ .

*Proof.* Since our coactions satisfy the Podleś conditions, Lemma 3.19 is the special case  $X = C$  and  $Y = D$  of Lemma 3.20. Hence it is enough to verify Lemma 3.20.

Let  $(\bar{\alpha}, \bar{\beta})$  be a V-anti Heisenberg pair acting on a Hilbert space  $\mathcal{K}$ . The definition of  $\iota_C$  and the comodule property (2.16) for  $\gamma$  yield

$$(\iota_C \otimes \bar{\alpha})\gamma = ((\text{id}_C \otimes \alpha \otimes \bar{\alpha})(\gamma \otimes \text{id}_A)\gamma)_{134} = ((\text{id}_C \otimes (\alpha \otimes \bar{\alpha})\Delta_A)\gamma)_{134};$$

Similarly,

$$(\iota_D \otimes \bar{\beta})\delta = ((\text{id}_D \otimes (\beta \otimes \bar{\beta})\Delta_B)\delta)_{234}.$$

Now Proposition 3.15 yields

$$(3.21) \quad (\iota_C \otimes \bar{\alpha})\gamma(c) \cdot (\iota_D \otimes \bar{\beta})\delta(d) = (\iota_D \otimes \bar{\beta})\delta(d) \cdot (\iota_C \otimes \bar{\alpha})\gamma(c)$$

for all  $c \in C$ ,  $d \in D$ .

Since  $\bar{\alpha}(A) \cdot \mathbb{K}(\mathcal{K}) = \mathbb{K}(\mathcal{K})$ , our assumption  $\gamma(X) \cdot (1_C \otimes A) = X \otimes A$  gives

$$\begin{aligned} ((\iota_C \otimes \bar{\alpha})\gamma(X)) \cdot \mathbb{K}(\mathcal{K})_4 &= (\iota_C \otimes \bar{\alpha})(\gamma(X) \cdot (1_C \otimes A)) \cdot \mathbb{K}(\mathcal{K})_4 \\ &= (\iota_C(X) \otimes \bar{\alpha}(A)) \cdot (1_C \otimes 1_{\mathcal{H}} \otimes \mathbb{K}(\mathcal{K}))_{134} = \iota_C(C) \otimes \mathbb{K}(\mathcal{K}). \end{aligned}$$

Similarly,  $\bar{\beta}(B) \cdot \mathbb{K}(\mathcal{K}) = \mathbb{K}(\mathcal{K})$  and our Podleś condition for  $\delta$  give

$$((\iota_D \otimes \bar{\beta})\delta(Y)) \cdot \mathbb{K}(\mathcal{K})_4 = \iota_D(Y) \otimes \mathbb{K}(\mathcal{K}).$$

Equation (3.21) gives

$$(\iota_C \otimes \bar{\alpha})\gamma(X) \cdot (\iota_D \otimes \bar{\beta})\delta(Y) = (\iota_D \otimes \bar{\beta})\delta(Y) \cdot (\iota_C \otimes \bar{\alpha})\gamma(X).$$

Multiplying this equation on the right with  $1_{C \otimes D \otimes \mathcal{H}} \otimes \mathbb{K}(\mathcal{K})$  and using the computations above to simplify, we get

$$(\iota_C(X) \cdot \iota_D(Y)) \otimes \mathbb{K}(\mathcal{K}) = (\iota_D(X) \cdot \iota_C(Y)) \otimes \mathbb{K}(\mathcal{K}).$$

Applying a state  $\omega$  on  $\mathbb{K}(\mathcal{K})$  to this equation gives  $\iota_C(X) \cdot \iota_D(Y) = \iota_D(X) \cdot \iota_C(Y)$  as desired.  $\square$

The following observation is useful to study slice maps on  $C \boxtimes_V D$ .

**Lemma 3.22.** *In the situation of the previous lemma,*

$$(3.23) \quad \iota_C(X) \cdot \iota_D(Y) \cdot \mathbb{K}(\mathcal{H})_3 = X \otimes Y \otimes \mathbb{K}(\mathcal{H}),$$

where the right hand side means the closed linear span of  $x \otimes y \otimes z$  with  $x \in X$ ,  $y \in Y$ ,  $z \in \mathbb{K}(\mathcal{H})$ . In particular,  $(C \boxtimes_V D) \cdot \mathbb{K}(\mathcal{H})_3 = C \otimes D \otimes \mathbb{K}(\mathcal{H})$ .

*Proof.* Since  $\mathbb{K}(\mathcal{H}) = \beta(B) \cdot \mathbb{K}(\mathcal{H})$ , we may compute

$$\begin{aligned} \iota_D(Y) \cdot \mathbb{K}(\mathcal{H})_3 &= ((\text{id}_D \otimes \beta)(\delta(Y) \cdot (1_D \otimes B)))_{23} \mathbb{K}(\mathcal{H})_3 \\ &= (Y \otimes \beta(B) \cdot \mathbb{K}(\mathcal{H}))_{23} = \mathbb{K}(\mathcal{H})_3 \cdot Y_2. \end{aligned}$$

Here  $Y_2$  and  $\mathbb{K}(\mathcal{H})_3$  mean  $Y$  and  $\mathbb{K}(\mathcal{H})$  in the second and third leg, respectively. A similar computation for  $\iota_C(X)$  using  $\mathbb{K}(\mathcal{H}) = \alpha(A) \cdot \mathbb{K}(\mathcal{H})$  now gives (3.23).  $\square$

We will consider interesting examples of this construction in Section 6. Here we only mention a trivial example:

*Example 3.24.* Assume the coaction  $\gamma$  is trivial. Then  $\gamma(c)_{13} = c \otimes 1 \otimes 1$ , so that we get  $C \boxtimes_V D \cong C \otimes D$ , embedded into  $\mathcal{M}(C \otimes D \otimes \mathbb{K}(\mathcal{H}))$  via  $\text{id}_C \otimes (\text{id}_D \otimes \beta)\delta$ . We get the same conclusion if  $\delta$  is trivial.

**Lemma 3.25.** *Let  $C_0$  be a  $C^*$ -algebra with trivial  $\mathbb{G}$ -coaction and equip  $C_0 \otimes C$  with the coaction  $\text{id}_{C_0} \otimes \gamma$ . Then*

$$(3.26) \quad (C_0 \otimes C) \boxtimes_V D = C_0 \otimes (C \boxtimes_V D).$$

*A similar statement holds in the second variable.*

*Proof.* The map  $\iota_{C_0 \otimes C}$  is  $\text{id}_{C_0} \otimes \iota_C$ .  $\square$

## 4. HILBERT SPACE REPRESENTATION OF THE TWISTED TENSOR PRODUCT

Let  $\mathbb{G} = (A, \Delta_A)$ ,  $\mathbb{H} = (B, \Delta_B)$ ,  $V \in \mathcal{U}(\hat{A} \otimes \hat{B})$ ,  $(C, \gamma)$  and  $(D, \delta)$  be as before, so that the twisted tensor product  $C \boxtimes_V D$  is defined. We are now going to construct a faithful Hilbert space representation of  $C \boxtimes_V D$  using covariant Hilbert space representations of  $(C, \gamma)$  and  $(D, \delta)$ . This yields an alternative definition of  $C \boxtimes_V D$  and shows that  $C \boxtimes_V D$  does not depend on the Heisenberg pair used in its construction.

Our new construction uses faithful covariant representations  $(\varphi, U^\mathcal{H})$  of  $(C, \gamma, A)$  and  $(\psi, U^\mathcal{K})$  of  $(D, \delta, B)$  on Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$ , respectively. (We will show below that such faithful covariant representations always exist.)

The bicharacter  $V$  and the corepresentations provide a unitary operator  $Z$  on  $\mathcal{H} \otimes \mathcal{K}$  as follows:

**Theorem 4.1.** *Let  $U^\mathcal{H} \in \mathcal{U}(\mathcal{H} \otimes A)$  and  $U^\mathcal{K} \in \mathcal{U}(\mathcal{K} \otimes B)$  be corepresentations of  $\mathbb{G}$  and  $\mathbb{H}$ , respectively. Then there is a unique unitary  $Z \in \mathcal{U}(\mathcal{H} \otimes \mathcal{K})$  that satisfies*

$$(4.2) \quad U_{1\alpha}^\mathcal{H} U_{2\beta}^\mathcal{K} Z_{12} = U_{2\beta}^\mathcal{K} U_{1\alpha}^\mathcal{H} \quad \text{in } \mathcal{U}(\mathcal{H} \otimes \mathcal{K} \otimes \mathcal{L})$$

for any  $V$ -Heisenberg pair  $(\alpha, \beta)$  acting on some Hilbert space  $\mathcal{L}$ .

With this unitary  $Z$ , define representations  $\varphi_1$  and  $\tilde{\psi}_2$  of  $C$  and  $D$  on  $\mathcal{H} \otimes \mathcal{K}$  by

$$\begin{aligned} \varphi_1(c) &:= \varphi(c) \otimes 1_\mathcal{K}, \\ \tilde{\psi}_2(d) &:= Z(1_\mathcal{H} \otimes \psi(d))Z^*. \end{aligned}$$

We can now formulate the main theorem of this section:

**Theorem 4.3.** *Let  $(\varphi, U^\mathcal{H})$  and  $(\psi, U^\mathcal{K})$  be faithful covariant representations of  $(C, \gamma, A)$  and  $(D, \delta, B)$  on Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$ , respectively. Construct  $\varphi_1$  and  $\tilde{\psi}_2$  as above. Then there is a unique faithful representation  $\rho: C \boxtimes_V D \rightarrow \mathbb{B}(\mathcal{H} \otimes \mathcal{K})$  with  $\rho \circ \iota_C = \varphi_1$  and  $\rho \circ \iota_D = \tilde{\psi}_2$ .*

*Example 4.4.* If  $V = 1$ , then we may take  $Z = 1$ . Thus  $\tilde{\psi}_2 = \psi_2$  and the crossed product is simply the minimal tensor product  $C \otimes D$ .

In the rest of this section, we prove the claims above and use the main theorem to show that the twisted tensor product does not depend on auxiliary choices. First we construct faithful covariant representations:

*Example 4.5.* Let  $\varphi_0: C \rightarrow \mathbb{B}(\mathcal{H}_0)$  be any faithful Hilbert space representation. Let  $(\pi, \hat{\pi})$  be a faithful  $\mathbb{G}$ -Heisenberg pair on a Hilbert space  $\mathcal{H}_\pi$ ; this exists because of Example 3.9. Let  $\mathcal{H} := \mathcal{H}_0 \otimes \mathcal{H}_\pi$  and identify  $\mathbb{K}(\mathcal{H}) \cong \mathbb{K}(\mathcal{H}_0) \otimes \mathbb{K}(\mathcal{H}_\pi)$ . The unitary  $U := 1_{\mathcal{H}_0} \otimes W_{\hat{\pi}2}^A \in \mathcal{U}(\mathbb{K}(\mathcal{H}) \otimes A)$  is a corepresentation; since  $\varphi_0$ ,  $\pi$  and  $\gamma$  are faithful morphisms,  $\varphi := (\varphi_0 \otimes \pi) \circ \gamma: C \rightarrow \mathbb{B}(\mathcal{H})$  is a faithful representation. The following computation in  $\mathcal{M}(C \otimes \mathbb{K}(\mathcal{H}_\pi) \otimes A)$  implies the covariance condition for  $(\varphi, U)$ :

$$((\text{id}_C \otimes \pi)\gamma \otimes \text{id}_A)\gamma(c) = (\text{id}_C \otimes (\pi \otimes \text{id}_A)\Delta_A)\gamma(c) = (W_{\hat{\pi}3}^A)(\gamma(c) \otimes 1_A)(W_{\hat{\pi}3}^A)^*$$

for all  $c \in C$ , where we used (2.16) and Lemma 3.11.3 with  $B = \hat{A}$  and  $\Delta_R = \Delta_A$ .

Now we prove Theorem 4.1. The uniqueness of  $Z$  is clear from

$$Z_{12} = (U_{2\beta}^\mathcal{K})^* (U_{1\alpha}^\mathcal{H})^* U_{2\beta}^\mathcal{K} U_{1\alpha}^\mathcal{H}.$$

Existence means that the operator on the right acts identically on the third leg and does not depend on the Heisenberg pair. The quickest way to prove this uses universal quantum groups to turn corepresentations into representations.

The universal quantum group  $\mathbb{G}^u := (A^u, \Delta_{A^u})$  associated to  $\mathbb{G} = (A, \Delta_A)$  is introduced in [12]. By construction, there is a universal (left) corepresentation

$\widetilde{\mathcal{W}}^A \in \mathcal{U}(\hat{A}^u \otimes A)$ ; its universal property gives a unique representation  $\rho_1: \hat{A}^u \rightarrow \mathbb{B}(\mathcal{H})$  with  $(\rho_1 \otimes \text{id}_A)\widetilde{\mathcal{W}}^A = U^{\mathcal{H}}$  because  $U^{\mathcal{H}}$  is a corepresentation. Similarly, there is a unique representation  $\rho_2: \hat{B}^u \rightarrow \mathbb{B}(\mathcal{K})$  with  $(\rho_2 \otimes \text{id}_B)\widetilde{\mathcal{W}}^B = U^{\mathcal{K}}$ .

[8, Proposition 4.7] shows that any bicharacter  $V \in \mathcal{U}(\hat{A} \otimes \hat{B})$  lifts uniquely to a bicharacter  $V^u \in \mathcal{U}(\hat{A}^u \otimes \hat{B}^u)$ . We claim that

$$(4.6) \quad Z := (\rho_1 \otimes \rho_2)(V^u)^* \in \mathbb{B}(\mathcal{H} \otimes \mathcal{K})$$

verifies (4.2) (for any V-Heisenberg pair  $(\alpha, \beta)$ ). (Our formulation of Theorem 4.1 highlights the property of the operator  $(\rho_1 \otimes \rho_2)(V^u)^*$  that is crucial for the proof of Theorem 4.3, and it avoids universal quantum groups.)

We will actually prove

$$(4.7) \quad \widetilde{\mathcal{W}}_{1\alpha}^A \widetilde{\mathcal{W}}_{2\beta}^B = \widetilde{\mathcal{W}}_{2\beta}^B \widetilde{\mathcal{W}}_{1\alpha}^A V_{12}^u \quad \text{in } \mathcal{U}(\hat{A}^u \otimes \hat{B}^u \otimes \mathbb{K}(\mathcal{L}))$$

for any V-Heisenberg pair  $(\alpha, \beta)$ . Applying  $\rho_1$  and  $\rho_2$  to the first two legs then gives (4.2) because  $(\rho_1 \otimes \text{id}_A)\widetilde{\mathcal{W}}^A = U^{\mathcal{H}}$  and  $(\rho_2 \otimes \text{id}_B)\widetilde{\mathcal{W}}^B = U^{\mathcal{K}}$ .

When we apply the reducing morphisms  $\Lambda_{\hat{A}}: \hat{A}^u \rightarrow \hat{A}$  and  $\Lambda_{\hat{B}}: \hat{B}^u \rightarrow \hat{B}$  to the first two legs in (4.7), we get  $W_{1\alpha}^A W_{2\beta}^B = W_{2\beta}^B W_{1\alpha}^A V_{12}$ , which is exactly the definition of a Heisenberg pair (see Definition 3.1). A routine computation shows that

$$T := (\widetilde{\mathcal{W}}_{1\alpha}^A)^* (\widetilde{\mathcal{W}}_{2\beta}^B)^* \widetilde{\mathcal{W}}_{1\alpha}^A \widetilde{\mathcal{W}}_{2\beta}^B \in \mathcal{U}(\hat{A}^u \otimes \hat{B}^u \otimes \mathbb{K}(\mathcal{L}))$$

is a character in the first two legs, that is,  $(\hat{\Delta}_{A^u} \otimes \text{id}_{\hat{B}^u} \otimes \text{id}_{\mathcal{L}})T = T_{234}T_{134}$  and  $(\text{id}_{\hat{A}^u} \otimes \hat{\Delta}_{B^u} \otimes \text{id}_{\mathcal{L}})T = T_{124}T_{134}$ . Thus  $T$  and  $V_{12}^u$  are two bicharacters in  $\mathcal{U}(\hat{A}^u \otimes \hat{B}^u \otimes \mathbb{K}(\mathcal{L}))$  that both lift the bicharacter  $V_{12}$  in  $\mathcal{U}(\hat{A} \otimes \hat{B} \otimes \mathbb{K}(\mathcal{L}))$ . Using [8, Lemma 4.6] twice, we get that any such bicharacter has a unique lifting. Thus  $T = V_{12}^u$  as asserted. This finishes the proof of Theorem 4.1.

Now we come to the proof of Theorem 4.3. The Hilbert space representation

$$\varphi \otimes \psi \otimes \text{id}: A \otimes B \otimes \mathbb{K}(\mathcal{L}) \rightarrow \mathbb{B}(\mathcal{H} \otimes \mathcal{K} \otimes \mathcal{L})$$

is faithful because  $\varphi$  and  $\psi$  are faithful. Hence the pair of representations

$$\begin{aligned} (\varphi \otimes \alpha)\gamma_{13}: C &\rightarrow \mathbb{B}(\mathcal{H} \otimes \mathcal{K} \otimes \mathcal{L}) \\ (\psi \otimes \beta)\delta_{23}: D &\rightarrow \mathbb{B}(\mathcal{H} \otimes \mathcal{K} \otimes \mathcal{L}) \end{aligned}$$

of  $C$  and  $D$  gives a faithful representation of  $C \boxtimes_{\mathbb{V}} D$ ; that is, there is a unique faithful representation of  $C \boxtimes_{\mathbb{V}} D$  that gives the above two representations when composed with  $\iota_C$  and  $\iota_D$ .

**Lemma 4.8.** *The pair of representations  $(\varphi_1, \text{Ad}_{Z_{12}} \circ \psi_2)$  of  $(C, D)$  on  $\mathcal{H} \otimes \mathcal{K} \otimes \mathcal{L}$  is unitarily equivalent to the pair  $((\varphi \otimes \alpha)\gamma_{13}, (\psi \otimes \beta)\delta_{23})$  on the same Hilbert space through conjugation by the unitary  $U_{1\alpha}^{\mathcal{H}} U_{2\beta}^{\mathcal{K}}$ .*

*Proof.* We must prove

$$\begin{aligned} U_{1\alpha}^{\mathcal{H}} U_{2\beta}^{\mathcal{K}} (\varphi(c) \otimes 1_{\mathcal{K}} \otimes 1_{\mathcal{L}}) (U_{2\beta}^{\mathcal{K}})^* (U_{1\alpha}^{\mathcal{H}})^* &= (\varphi \otimes \alpha)\gamma_{13}(c), \\ U_{1\alpha}^{\mathcal{H}} U_{2\beta}^{\mathcal{K}} Z_{12} (1_{\mathcal{H}} \otimes \psi(d) \otimes 1_{\mathcal{L}}) Z_{12}^* (U_{2\beta}^{\mathcal{K}})^* (U_{1\alpha}^{\mathcal{H}})^* &= (\psi \otimes \beta)\delta_{23}(d) \end{aligned}$$

for all  $c \in C$ ,  $d \in D$ . To check the first equality, we use first that  $U_{2\beta}^{\mathcal{K}}$  commutes with  $\varphi(c)_1$  because both act on different legs, and secondly the covariance condition (2.23) for  $(\varphi, U^{\mathcal{H}})$  with  $\alpha$  applied to the leg  $A$ :

$$U_{1\alpha}^{\mathcal{H}} U_{2\beta}^{\mathcal{K}} (\varphi(c) \otimes 1_{\mathcal{K}} \otimes 1_{\mathcal{L}}) (U_{2\beta}^{\mathcal{K}})^* (U_{1\alpha}^{\mathcal{H}})^* = U_{1\alpha}^{\mathcal{H}} (\varphi(c) \otimes 1_{\mathcal{K}} \otimes 1_{\mathcal{L}}) (U_{1\alpha}^{\mathcal{H}})^* = (\varphi \otimes \alpha)\gamma(c)_{13}.$$

To check the second equality, we first use (4.2); secondly, that  $U_{1\alpha}^{\mathcal{H}}$  and  $\psi(d)_2$  act in different legs to commute them; and thirdly the covariance condition (2.23) for

$(\psi, U^K)$  with  $\beta$  applied to the leg  $B$ :

$$\begin{aligned} U_{1\alpha}^{\mathcal{H}} U_{2\beta}^{\mathcal{K}} Z_{12} (1_{\mathcal{H}} \otimes \psi(d) \otimes 1_{\mathcal{L}}) Z_{12}^* (U_{2\beta}^{\mathcal{K}})^* (U_{1\alpha}^{\mathcal{H}})^* &= U_{2\beta}^{\mathcal{K}} U_{1\alpha}^{\mathcal{H}} \psi(d)_2 (U_{1\alpha}^{\mathcal{H}})^* (U_{2\beta}^{\mathcal{K}})^* \\ &= U_{2\beta}^{\mathcal{K}} \psi(d)_2 (U_{2\beta}^{\mathcal{K}})^* = (\psi \otimes \beta) \delta(d)_{23}. \quad \square \end{aligned}$$

We remarked above that the pair of representations  $((\varphi \otimes \alpha) \gamma_{13}, (\psi \otimes \beta) \delta_{23})$  generates a faithful representation of  $C \boxtimes_V D$ . Lemma 4.8 shows that this representation is unitarily equivalent to another representation that restricts to  $\varphi_1 \otimes 1_{\mathcal{L}}$  and  $\text{Ad}_{Z_{12}} \circ \psi_2 = \psi_2 \otimes 1_{\mathcal{L}}$  on  $C$  and  $D$ , respectively. The latter representation is  $\rho \otimes 1_{\mathcal{L}}$  for a faithful representation of  $C \boxtimes_V D$  on  $\mathcal{H} \otimes \mathcal{K}$ . This is the faithful representation whose existence is asserted in Theorem 4.3. Uniqueness is clear because  $C \boxtimes_V D = \iota_C(C) \cdot \iota_D(D)$ . This finishes the proof of Theorem 4.3.

**Theorem 4.9.** *In the notation of Theorem 4.3, the subspace*

$$C \tilde{\boxtimes}_V D := \varphi_1(C) \cdot \tilde{\psi}_2(D) \subseteq \mathbb{B}(\mathcal{H} \otimes \mathcal{K})$$

*is a  $C^*$ -subalgebra and  $(C \tilde{\boxtimes}_V D, \varphi_1, \tilde{\psi}_2)$  is a crossed product of  $C$  and  $D$ . Up to equivalence of crossed products, it does not depend on  $(\varphi, U^{\mathcal{H}})$  and  $(\psi, U^{\mathcal{K}})$ .*

*The crossed product  $(C \boxtimes_V D, \iota_C, \iota_D)$  is equivalent to  $(C \tilde{\boxtimes}_V D, \varphi_1, \tilde{\psi}_2)$  and, up to equivalence of crossed products, does not depend on the Heisenberg pair  $(\alpha, \beta)$ .*

*Proof.* Since  $C \tilde{\boxtimes}_V D = \rho(C \boxtimes_V D)$  and  $\rho \circ \iota_C = \varphi_1$ ,  $\rho \circ \iota_D = \tilde{\psi}_2$ , by Theorem 4.3,  $C \tilde{\boxtimes}_V D$  is a  $C^*$ -algebra,  $(C \tilde{\boxtimes}_V D, \varphi_1, \tilde{\psi}_2)$  is a crossed product of  $C$  and  $D$ , and it is equivalent to the crossed product  $(C \boxtimes_V D, \iota_C, \iota_D)$ .

Since the unitary  $Z$  is the same for all Heisenberg pairs  $(\alpha, \beta)$ , the crossed product  $(C \tilde{\boxtimes}_V D, \varphi_1, \tilde{\psi}_2)$  does not depend on  $(\alpha, \beta)$ ; hence up to equivalence  $(C \boxtimes_V D, \iota_C, \iota_D)$  does not depend on  $(\alpha, \beta)$ . And since  $(C \boxtimes_V D, \iota_C, \iota_D)$  does not depend on  $(\varphi, U^{\mathcal{H}})$  and  $(\psi, U^{\mathcal{K}})$ , neither does  $(C \tilde{\boxtimes}_V D, \varphi_1, \tilde{\psi}_2)$ , up to equivalence.  $\square$

As a special case Theorem 4.3, the usual spatial tensor product  $C \otimes D$  does not depend on the chosen faithful representations of  $C$  and  $D$ . But we do not reprove this classical result. Rather, we reduce analogous statements for noncommutative tensor products to this case by embedding the latter into commutative tensor products with more factors.

## 5. PROPERTIES OF THE TWISTED TENSOR PRODUCT

In this section, we establish several functoriality properties of the twisted tensor product. We also discuss exactness for equivariantly semi-split extensions and invariance under Morita–Rieffel equivalence, which gives a result about cocycle conjugacy.

We begin with an easy symmetry property:

**Proposition 5.1.** *The crossed products  $(C \boxtimes_V D, \iota_C, \iota_D)$  and  $(D \boxtimes_{\widehat{V}} C, \iota'_D, \iota'_C)$  are canonically isomorphic.*

*Proof.* Let  $(U^{\mathcal{H}}, \varphi)$  and  $(U^{\mathcal{K}}, \psi)$  be faithful covariant representations of  $C$  and  $D$  on Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$ , respectively. Theorem 4.9 yields

$$\begin{aligned} (C \boxtimes_V D, \iota_C, \iota_D) &\cong (C \tilde{\boxtimes}_V D, \varphi_1, \tilde{\psi}_2), \\ (D \boxtimes_{\widehat{V}} C, \iota'_D, \iota'_C) &\cong (D \tilde{\boxtimes}_{\widehat{V}} C, \psi_1, \tilde{\varphi}_2) \end{aligned}$$

with  $C \tilde{\boxtimes}_V D \subseteq \mathbb{B}(\mathcal{H} \otimes \mathcal{K})$ ,  $D \tilde{\boxtimes}_{\widehat{V}} C \subseteq \mathbb{B}(\mathcal{K} \otimes \mathcal{H})$ ; here  $\psi_1(d) := (\psi(d) \otimes 1_{\mathcal{H}})$  and  $\tilde{\varphi}_2(c) := \hat{Z}(1_{\mathcal{K}} \otimes \psi(c)) \hat{Z}^*$ , where  $Z$  satisfies (4.2) and  $\hat{Z} = \Sigma Z^* \Sigma$ . The pair of representations  $(\varphi_1, \tilde{\psi}_2)$  of  $(C, D)$  on  $\mathcal{H} \otimes \mathcal{K}$  is unitarily equivalent to the pair of representations  $(\psi_1, \tilde{\varphi}_2)$  on  $\mathcal{K} \otimes \mathcal{H}$  via the unitary  $\Sigma Z^*$ .  $\square$

**5.1. Functoriality for quantum group morphisms.** Let  $\mathbb{G} = (A, \Delta_A)$ ,  $\mathbb{H} = (B, \Delta_B)$ ,  $\mathbb{G}_2 = (A_2, \Delta_{A_2})$  and  $\mathbb{H}_2 = (B_2, \Delta_{B_2})$  be quantum groups. Let  $f: \mathbb{G} \rightarrow \mathbb{G}_2$  and  $g: \mathbb{H} \rightarrow \mathbb{H}_2$  be quantum group morphisms in the sense of the several equivalent descriptions in Theorem 2.29.

Let  $V_2 \in \mathcal{U}(\hat{A}_2 \otimes \hat{B}_2)$  be a bicharacter. We may view  $V_2$  as a quantum group morphism  $V'_2: \mathbb{G}_2 \rightarrow \hat{\mathbb{H}}_2$ . Composing this with the given quantum group morphisms  $f: \mathbb{G} \rightarrow \mathbb{G}_2$  and the dual  $\hat{g}: \hat{\mathbb{H}}_2 \rightarrow \hat{\mathbb{H}}$ , we get a quantum group morphism  $V' := \hat{g} \circ V'_2 \circ f: \mathbb{G} \rightarrow \hat{\mathbb{H}}$ , which we view as a bicharacter  $V \in \mathcal{U}(\hat{A} \otimes \hat{B})$ .

Let  $(C, \gamma)$  and  $(D, \delta)$  be a  $\mathbb{G}$ - $C^*$ -algebra and an  $\mathbb{H}$ - $C^*$ -algebra, respectively. The description of  $f$  in Theorem 2.29(4) is as a functor from the category of  $\mathbb{G}$ - $C^*$ -algebras to  $\mathbb{G}_2$ - $C^*$ -algebras that does not change the underlying  $C^*$ -algebra. In particular, this functor maps  $\gamma$  to a continuous  $\mathbb{G}_2$ -coaction  $\gamma_2: C \rightarrow C \otimes A_2$  on  $C$ . Similarly,  $g$  maps  $\delta$  to a continuous  $\mathbb{H}_2$ -coaction  $\delta_2: D \rightarrow D \otimes B_2$  on  $D$ .

**Theorem 5.2.** *In the situation above, the crossed products  $(C, \gamma_2) \boxtimes_{V_2} (D, \gamma_2)$  and  $(C, \gamma) \boxtimes_V (D, \gamma)$  of  $C$  and  $D$  are equivalent.*

*Proof.* Let  $(\varphi, U^{\mathcal{H}})$  be a  $\mathbb{G}$ -covariant representation of  $(C, \gamma)$  on  $\mathcal{H}$  and let  $(\psi, U^{\mathcal{K}})$  be a  $\mathbb{H}$ -covariant representation of  $(D, \delta)$  on  $\mathcal{K}$ .

The quantum group morphism  $f$  turns  $U^{\mathcal{H}}$  into a corepresentation  $U_2^{\mathcal{H}}$  of  $\mathbb{G}_2$  on  $\mathcal{H}$ . This is asserted in [8, Proposition 6.5]. Since the quick proof given in [8] only works for corepresentations that induce a continuous coaction on  $\mathbb{K}(\mathcal{H})$ , which is not automatic, we give a different proof here using universal quantum groups.

We may view the quantum group morphism  $f$  as a Hopf morphism  $\hat{f}: \hat{A}_2^u \rightarrow \hat{A}^u$  between the duals of the associated universal  $C^*$ -algebras by [8, Theorem 4.8]. By the universal property,  $U^{\mathcal{H}}$  is equivalent to a representation of  $\hat{A}^u$  on  $\mathcal{H}$ . Composing this with  $\hat{f}$  gives a representation of  $\hat{A}_2^u$ , which is equivalent to the desired corepresentation  $U_2^{\mathcal{H}}$  of  $\mathbb{G}_2$  on  $\mathcal{H}$ .

This operation on the level of corepresentations is compatible with the map  $\gamma \mapsto \gamma_2$  on coactions in the sense that  $(\varphi, U_2^{\mathcal{H}})$  is a  $\mathbb{G}_2$ -covariant representation of  $(C, \gamma_2)$ .

Similarly,  $g$  turns  $U^{\mathcal{K}}$  into a corepresentation  $U_2^{\mathcal{K}}$  of  $\mathbb{H}_2$  on  $\mathcal{K}$ , and  $(\psi, U_2^{\mathcal{K}})$  is a covariant representation of  $(D, \delta_2)$ .

The bicharacters  $V \in \mathcal{U}(\hat{A} \otimes \hat{B})$  and  $V_2 \in \mathcal{U}(\hat{A}_2 \otimes \hat{B}_2)$  lift uniquely to a bicharacters  $V^u \in \mathcal{U}(\hat{A}^u \otimes \hat{B}^u)$  and  $V_2^u \in \mathcal{U}(\hat{A}_2^u \otimes \hat{B}_2^u)$  by [8, Proposition 4.7]. The bijection between bicharacters and quantum group morphisms is defined in such a way that  $V^u = (\hat{f} \otimes \hat{g})(V_2^u)$ . Equation 4.6 then shows that the unitaries  $Z$  on  $\mathcal{H} \otimes \mathcal{K}$  that are used to construct the twisted tensor products with respect to  $V$  and  $V_2$  are the same.

Now Theorem 4.3 yields the desired equivalence of crossed products because both are faithfully represented by the same  $C^*$ -algebra  $\varphi(C) \cdot Z\psi(D)Z^*$  on  $\mathcal{H} \otimes \mathcal{K}$ .  $\square$

The following special cases of Theorem 5.2 are particularly noteworthy.

*Example 5.3.* Let  $\mathbb{G}_2 = \hat{\mathbb{H}}$ ,  $\mathbb{H}_2 = \mathbb{H}$ , let  $g = \text{id}: \mathbb{H} \rightarrow \mathbb{H}_2$  and let  $f: \mathbb{G} \rightarrow \mathbb{G}_2 = \hat{\mathbb{H}}$  be the bicharacter  $V$  itself, viewed a quantum group morphism as above. Let  $V_2 = \widehat{W}^B$  be the reduced bicharacter of  $\hat{\mathbb{H}}$ . Then

$$(C, \gamma) \boxtimes_V (D, \delta) \cong (C, \gamma_2) \boxtimes_{\widehat{W}^B} (D, \delta),$$

where  $\gamma_2: C \rightarrow C \otimes \hat{B}$  is the  $\hat{\mathbb{H}}$ -coaction associated to  $\gamma$  by the quantum group morphism  $f$  corresponding to  $V$ .

This is a special case of Theorem 5.2 because the bicharacter  $\widehat{W}^B$  describes the identity morphism on the quantum group  $\hat{\mathbb{H}}$ . The composition of this with  $f$

gives again  $f$ , so that the bicharacter  $V$  that we get from  $V_2 = \widehat{W}^B$  by the above construction is indeed the given one.

*Example 5.4.* Let  $\mathbb{G}_2 = \mathbb{G}$ ,  $\mathbb{H}_2 = \hat{\mathbb{G}}$ , let  $f = \text{id}: \mathbb{G}_2 \rightarrow \mathbb{G}$  and let  $g: \mathbb{H} \rightarrow \mathbb{H}_2 = \hat{\mathbb{G}}$  be the dual of the morphism  $\mathbb{G} \rightarrow \hat{\mathbb{H}}_2$  associated to the bicharacter  $V$ . Let  $V_2 = W^A$  be the reduced bicharacter of  $\hat{\mathbb{H}}$ . Then

$$(C, \gamma) \boxtimes_V (D, \delta) \cong (C, \gamma) \boxtimes_{W^A} (D, \delta_2),$$

where  $\delta_2: D \rightarrow D \otimes \hat{A}$  is the  $\hat{\mathbb{G}}$ -coaction associated to  $\delta$  by the quantum group morphism  $g$ .

The last example reduce the twisted tensor product  $\boxtimes_V$  for an arbitrary bicharacter to the special case  $\mathbb{H} = \hat{\mathbb{G}}$  and  $V = W^A$ .

**5.2. Functoriality for various kinds of maps.**  $C^*$ -algebras may be turned into a category using several types of maps:

- morphisms (nondegenerate  $*$ -homomorphisms  $C_1 \rightarrow \mathcal{M}(C_2)$ );
- proper morphisms (nondegenerate  $*$ -homomorphisms  $C_1 \rightarrow C_2$ );
- possibly degenerate  $*$ -homomorphisms  $C_1 \rightarrow C_2$ ;
- completely positive maps  $C_1 \rightarrow C_2$ ;
- completely positive contractions  $C_1 \rightarrow C_2$ ;
- completely contractive maps  $C_1 \rightarrow C_2$ ;
- completely bounded maps  $C_1 \rightarrow C_2$ .

It is well known that the minimal tensor product is functorial for such maps; that is, two “maps”  $f: C_1 \rightarrow C_2$  and  $g: D_1 \rightarrow D_2$  induce a “map”  $f \otimes g: C_1 \otimes D_1 \rightarrow C_2 \otimes D_2$ , which is determined by  $(f \otimes g)(c \otimes d) := f(c) \otimes g(d)$ . We claim that the tensor product  $\boxtimes_V$  is also functorial for all these kinds of “maps” in the following sense:

**Lemma 5.5.** *If  $f: (C_1, \gamma_1) \rightarrow (C_2, \gamma_2)$  is a  $\mathbb{G}$ -equivariant “map” and  $g: (D_1, \delta_1) \rightarrow (D_2, \delta_2)$  is an  $\mathbb{H}$ -equivariant “map,” then there is a unique “map”*

$$f \boxtimes_V g: C_1 \boxtimes_V D_1 \rightarrow C_2 \boxtimes_V D_2, \quad \iota_{C_1}(c) \cdot \iota_{D_1}(d) \mapsto \iota_{C_2}(f(c)) \cdot \iota_{D_2}(g(d)),$$

and  $(f, g) \mapsto f \boxtimes_V g$  is an additive bifunctor.

The notion of equivariance for possibly degenerate  $*$ -homomorphisms or completely bounded maps is defined as in [1, Definition 1.8]. The multiplier algebra is not functorial for such maps, but the comultiplication morphism  $\gamma_1: C_1 \rightarrow \mathcal{M}(C_1 \otimes A)$  takes values in the smaller algebra

$$\tilde{\mathcal{M}}(C_1 \otimes A) := \{x \in \mathcal{M}(C_1 \otimes D) \mid x \cdot (1 \otimes D) \cup (1 \otimes D) \cdot x \subseteq C_1 \otimes D\}.$$

A completely bounded map  $f: C_1 \rightarrow C_2$  induces a completely bounded,  $\mathcal{M}(D)$ -bilinear map

$$f \otimes \text{id}_D: C_1 \otimes D \rightarrow C_2 \otimes D.$$

The map  $f \otimes \text{id}_D$  is completely positive or completely contractive if  $f$  is, and a  $*$ -homomorphism if  $f$  is. Any  $\mathcal{M}(D)$ -bilinear “map”  $C_1 \otimes D \rightarrow C_2 \otimes D$  extends uniquely to a  $\mathcal{M}(D)$ -bilinear “map”  $\tilde{\mathcal{M}}(C_1 \otimes D) \rightarrow \tilde{\mathcal{M}}(C_2 \otimes D)$ . We use this extension to make sense of the equivariance condition  $\delta \circ f = (f \otimes \text{id}_A) \circ \gamma$ .

*Proof of Lemma 5.5.* The uniqueness and hence the functoriality is clear because the linear span of  $\iota_{C_1}(c) \cdot \iota_{D_1}(d)$  with  $c \in C_1$ ,  $d \in D_1$  is dense in  $C_1 \boxtimes_V D_1$  and all types of “maps” we consider are bounded linear.

We remarked above that ordinary minimal  $C^*$ -tensor products are functorial for “maps”, that is, there is a well-defined  $\mathbb{B}(\mathcal{H})$ -linear “map”

$$f \otimes g \otimes \text{id}_{\mathbb{K}(\mathcal{H})}: C_1 \otimes D_1 \otimes \mathbb{K}(\mathcal{H}) \rightarrow C_2 \otimes D_2 \otimes \mathbb{K}(\mathcal{H}).$$



We may extend it to a “map”

$$(f, g)_* : \tilde{\mathcal{M}}_{\mathbb{K}(\mathcal{H})}(C_1 \otimes D_1 \otimes \mathbb{K}(\mathcal{H})) \rightarrow \tilde{\mathcal{M}}_{\mathbb{K}(\mathcal{H})}(C_2 \otimes D_2 \otimes \mathbb{K}(\mathcal{H})),$$

Lemma 3.22 implies  $C_i \boxtimes_V D_i \subseteq \tilde{\mathcal{M}}_{\mathbb{K}(\mathcal{H})}(C_i \otimes D_i \otimes \mathbb{K}(\mathcal{H}))$  for  $i = 1, 2$ .

We claim that the “map”  $(f, g)_*$  sends  $\iota_{C_1}(c)\iota_{D_1}(d)$  to  $\iota_{C_2}(f(c))\iota_{D_2}(g(d))$ ; hence it restricts to a “map”  $f \boxtimes_V g : C_1 \boxtimes_V D_1 \rightarrow C_2 \boxtimes_V D_2$  with the required property.

Now we look at some types of maps separately. First let  $f$  and  $g$  be equivariant morphisms; then  $f \otimes g \otimes \text{id}_{\mathbb{K}(\mathcal{H})}$  is a morphism, hence extends to a  $*$ -homomorphism between multiplier algebras. Since  $f$  and  $g$  are equivariant, this canonical extension maps  $\gamma_1(c)_{1\pi} \mapsto \gamma_2(c)_{1\pi}$  and  $\delta_1(c)_{2\pi} \mapsto \delta_2(c)_{2\pi}$ . Hence it maps  $\iota_{C_1}(c)\iota_{D_1}(d)$  to  $\iota_{C_2}(f(c))\iota_{D_2}(g(d))$  as needed. If  $f$  and  $g$  are proper morphisms, then  $\iota_{C_2}(f(c))\iota_{D_2}(g(d)) \in C_2 \boxtimes_V D_2$  for all  $c \in C_1, d \in D_1$ , so that  $f \boxtimes_V g$  is a proper morphism as well.

Now let  $f$  and  $g$  be completely bounded maps; this contains the remaining types as special cases. By definition,

$$\begin{aligned} (f \otimes g \otimes \text{id})(c \otimes 1_D \otimes x) \cdot (f \otimes g \otimes \text{id})(1_C \otimes d \otimes y) &= f(c) \otimes g(d) \otimes x \cdot y \\ &= (f \otimes g \otimes \text{id})(c \otimes d \otimes x \cdot y) \end{aligned}$$

for all  $c \in C_1, d \in D_1, x, y \in \mathbb{K}(\mathcal{H})$ . Since  $f \otimes g \otimes \text{id}$  is bounded linear, this implies the partial multiplicativity  $(f \otimes g \otimes \text{id})(x \cdot y) = (f \otimes \text{id})(x)_{13}(g \otimes \text{id})(y)_{23}$  if  $x \in \tilde{\mathcal{M}}(C_1 \otimes \mathbb{K}(\mathcal{H})), y \in \tilde{\mathcal{M}}(C_2 \otimes \mathbb{K}(\mathcal{H}))$ . In particular,

$$(f \otimes g \otimes \text{id}_{\mathbb{K}(\mathcal{H})})(\iota_{C_1}(c) \cdot \iota_{D_1}(d)) = (f \otimes g \otimes \text{id}_{\mathbb{K}(\mathcal{H})})(\iota_{C_1}(c)) \cdot (f \otimes g \otimes \text{id}_{\mathbb{K}(\mathcal{H})})(\iota_{D_1}(d))$$

for all  $c \in C_1, d \in D_1$ . Finally, the equivariance of  $f$  and  $g$  shows that the right hand side is  $\iota_{C_2}(f(c))\iota_{D_2}(g(d))$ .  $\square$

**Proposition 5.6.** *If  $f$  and  $g$  are injective morphisms or  $*$ -homomorphisms, then so is  $f \boxtimes_V g$ , and vice versa.*

*If  $f$  and  $g$  are surjective  $*$ -homomorphisms, then so is  $f \boxtimes_V g$ , and vice versa.*

*The map  $f \boxtimes_V g$  is invertible if and only if both  $f$  and  $g$  are invertible.*

*Proof.* If  $f$  and  $g$  are injective, so is  $f \otimes g \otimes \text{id}_{\mathbb{K}(\mathcal{H})}$ ; hence its extension to multipliers is injective, and so is the restriction to  $C_1 \boxtimes_V D_1$ . Conversely,  $(f \boxtimes_V g)(\iota_{C_1}(c)\iota_{D_1}(d))$  vanishes if  $f(c) = 0$  or  $g(d) = 0$ ; hence  $f$  and  $g$  are injective if  $f \boxtimes_V g$  is.

If  $f$  and  $g$  are surjective, then elements of the form  $(f \boxtimes_V g)(\iota_{C_1}(c)\iota_{D_1}(d)) = \iota_{C_2}(f(c))\iota_{D_2}(g(d))$  are linearly dense in  $C_2 \boxtimes_V D_2$ . Hence  $f \boxtimes_V g$  is surjective as well. Conversely, suppose that  $f \boxtimes_V g$  is surjective. Then

$$\iota_{C_2}(f(C_1))\iota_{D_2}(g(D_1)) \cdot \mathbb{K}(\mathcal{H})_3 = (C_2 \boxtimes_V D_2) \cdot \mathbb{K}(\mathcal{H})_3 = C_2 \otimes D_2 \otimes \mathbb{K}(\mathcal{H})$$

by Lemma 3.22. We also have  $\iota_{C_2}(f(C_1))\iota_{D_2}(g(D_1)) \cdot \mathbb{K}(\mathcal{H})_3 \subseteq f(C_1) \otimes g(D_1) \otimes \mathbb{K}(\mathcal{H})$ . Applying slice maps to  $C_2$  and  $D_2$ , we get  $f(C_1) = C_2$  and  $g(D_1) = D_2$ .  $\square$

We now use Proposition 5.6 for the equivariant embeddings  $\gamma : C \rightarrow C \otimes A$  and  $\delta : D \rightarrow D \otimes B$  provided in Lemma 2.18 to get an embedding

$$(C, \gamma) \boxtimes_V (D, \delta) \rightarrow C \otimes D \otimes (A, \Delta) \boxtimes_V (B, \Delta).$$

Thus we may describe  $(C, \gamma) \boxtimes_V (D, \delta)$  as the crossed product generated by the embeddings  $(\text{id} \otimes \iota_A)\gamma_{13}$  of  $C$  and  $(\text{id} \otimes \iota_B)\delta_{23}$  of  $D$  into  $C \otimes D \otimes (A, \Delta) \boxtimes_V (B, \Delta)$ .

This description is particularly useful if we can describe  $(A, \Delta) \boxtimes_V (B, \Delta)$  more explicitly. For instance, for the skew-commutative tensor product in Section 6.1, we get

$$C^*(\mathbb{Z}/2) \boxtimes_V C^*(\mathbb{Z}/2) \cong \mathbb{M}_2(\mathbb{C}),$$

generated by the anti-commuting involutions

$$g_1 := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \quad g_2 := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Thus the skew-commutative tensor product may be defined as the crossed product generated by the embeddings of  $C$  and  $D$  into  $\mathbb{M}_2(C \otimes D)$ , mapping  $c \mapsto c \otimes 1 \otimes 1$  for even  $c \in C$ ,  $c \mapsto c \otimes 1 \otimes g_1$  for odd  $c \in C$ ,  $d \mapsto 1 \otimes d \otimes 1$  for even  $d \in D$ ,  $d \mapsto 1 \otimes d \otimes g_2$  for odd  $d \in D$ .

### 5.3. Exactness on equivariantly semi-split extensions.

**Proposition 5.7.** *The functor  $\square \boxtimes_V D$  maps an extension  $C_1 \rightarrow C_2 \twoheadrightarrow C_3$  of  $\mathbb{G}$ - $C^*$ -algebras with a  $\mathbb{G}$ -equivariant completely bounded section to an extension of  $C^*$ -algebras with a completely bounded section. If the section  $C_3 \rightarrow C_2$  is an equivariant  $*$ -homomorphism, completely positive or completely contractive, then so is the induced section  $C_3 \boxtimes_V D \rightarrow C_2 \boxtimes_V D$ . Analogous statements hold for the functor  $C \boxtimes_V \square$ .*

*Proof.* We have  $C_1 \oplus C_3 \cong C_2$  in the additive category of  $\mathbb{G}$ -equivariant completely bounded maps, using the inclusion map  $C_1 \rightarrow C_2$  and the section  $s: C_3 \rightarrow C_2$ . Since  $\square \boxtimes_V D$  is an additive functor, this implies  $C_1 \boxtimes_V D \oplus C_3 \boxtimes_V D \cong C_2 \boxtimes_V D$  in the category of completely bounded maps. Thus

$$C_1 \boxtimes_V D \rightarrow C_2 \boxtimes_V D \rightarrow C_3 \boxtimes_V D$$

is an extension of  $C^*$ -algebras with  $s \boxtimes_V \text{id}_D$  as a completely bounded linear section. This section is again a  $*$ -homomorphism, completely contractive, or completely positive if  $s$  is so.  $\square$

The functor  $\square \boxtimes_V D$  cannot be exact for arbitrary extensions because this already fails for the commutative minimal tensor product. An  $\mathbb{H}$ - $C^*$ -algebra  $D$  deserves to be called “exact” if  $\square \boxtimes_V D$  is an exact functor for all  $\mathbb{G}$  and all bicharacters  $V \in \mathcal{U}(\hat{A} \hat{\otimes} \hat{B})$ . We leave this line of thought to future study.

**5.4. Functoriality for correspondences.** Next we want to show that  $\boxtimes_V$  is functorial for equivariant correspondences. Recall that a correspondence between two  $C^*$ -algebras  $C_1$  and  $C_2$  is a Hilbert  $C_2$ -module  $\mathcal{E}$  with a non-degenerate left  $C_1$ -action (by adjointable operators). In this section, we assume familiarity with Hilbert modules, see [7].

We want to show that a  $\mathbb{G}$ -equivariant correspondence  $\mathcal{E}: C_1 \rightarrow C_2$  and a  $\mathbb{H}$ -equivariant correspondence  $\mathcal{F}: D_1 \rightarrow D_2$  induce a correspondence

$$\mathcal{E} \boxtimes_V \mathcal{F}: C_1 \boxtimes_V D_1 \rightarrow C_2 \boxtimes_V D_2$$

with suitable functoriality properties, including compatibility with the composition of correspondences: given further equivariant correspondences  $\mathcal{E}_2: C_2 \rightarrow C_3$  and  $\mathcal{F}_2: D_2 \rightarrow D_3$ , there is a natural isomorphism of correspondences

$$(\mathcal{E} \otimes_{C_2} \mathcal{E}_2) \boxtimes_V (\mathcal{F} \otimes_{D_2} \mathcal{F}_2) \cong (\mathcal{E} \boxtimes_V \mathcal{F}) \otimes_{C_2 \boxtimes_V D_2} (\mathcal{E}_2 \boxtimes_V \mathcal{F}_2).$$

This also implies that if  $\mathcal{E}$  and  $\mathcal{F}$  are equivariant Morita–Rieffel equivalences (that is, full Hilbert bimodules), then  $\mathcal{E} \boxtimes_V \mathcal{F}$  is a Morita–Rieffel equivalence.

Quantum group coactions on Hilbert modules are defined by Baaĵ and Skandalis in [1, Definition 2.2], but without considering Podleś’ continuity condition. Therefore, we add one condition to our definition.

**Definition 5.8.** A  $\mathbb{G}$ -equivariant Hilbert module over a  $\mathbb{G}$ - $C^*$ -algebra  $(C, \gamma)$  is a Hilbert  $C$ -module  $\mathcal{E}$  with a coaction  $\epsilon: \mathcal{E} \rightarrow \tilde{\mathcal{M}}(\mathcal{E} \otimes A)$  with the following properties:

- (1)  $\epsilon(\xi)\gamma(c) = \epsilon(\xi c)$  for  $\xi \in \mathcal{E}$ ,  $c \in C$ ;
- (2)  $\gamma(\langle \xi, \eta \rangle_C) = \langle \epsilon(\xi), \epsilon(\eta) \rangle_{\tilde{\mathcal{M}}(C \otimes A)}$ ;
- (3)  $\epsilon(\mathcal{E}) \cdot (1 \otimes A) = \mathcal{E} \otimes A$ ;
- (4)  $(1 \otimes A) \cdot \epsilon(\mathcal{E}) = \mathcal{E} \otimes A$ ;
- (5)  $(\epsilon \otimes \text{id}_A)\epsilon = (\text{id}_{\mathcal{E}} \otimes \epsilon)\epsilon$ .

Here

$$\tilde{\mathcal{M}}(\mathcal{E} \otimes A) := \{T \in \mathbb{B}(C \otimes A, \mathcal{E} \otimes A) \mid (1_{\mathcal{E}} \otimes A)T \cup T(1_C \otimes A) \subseteq \mathcal{E} \otimes A\}.$$

Condition (5) uses canonical extensions of  $\epsilon \otimes \text{id}_A$  and  $\text{id}_A \otimes \epsilon$  to maps

$$\mathbb{B}(C \otimes A, \mathcal{E} \otimes A) \rightarrow \mathbb{B}(C \otimes A \otimes A, \mathcal{E} \otimes A \otimes A),$$

which are described in [1, Remarque 2.5]. The map  $\epsilon$  is automatically norm-isometric by [1, Proposition 2.4].

Since  $\gamma$  satisfies the Podleś condition and  $\epsilon(\mathcal{E}) = \epsilon(\mathcal{E}) \cdot \gamma(C)$ , our condition (3) is equivalent to  $\epsilon(\mathcal{E}) \cdot (C \otimes A) = \mathcal{E} \otimes A$ . Thus the conditions in [1, Definition 2.2] are equivalent to our conditions (1)–(3) and (5).

*Remark 5.9.* Our definition and the one by Baaj and Skandalis give the same definition for Hilbert bimodules and hence the same notion of equivariant Morita–Rieffel equivalence (provided the  $C^*$ -algebras involved carry continuous coactions).

A *Hilbert bimodule* between  $C_1$  and  $C_2$  is both a right Hilbert  $C_2$ -module and a left Hilbert  $C_1$ -module, such that the left and right module structures commute and the inner products satisfy  $\langle \xi, \eta \rangle_{C_1} \cdot \zeta = \xi \cdot \langle \eta, \zeta \rangle_{C_2}$  for all  $\xi, \eta, \zeta \in \mathcal{E}$ .

The left and the right Hilbert module structure both give the same multiplier space  $\tilde{\mathcal{M}}(\mathcal{E} \otimes A)$  because  $\mathbb{K}(\mathcal{H} \otimes A)$  maps  $\tilde{\mathcal{M}}(\mathcal{E} \otimes A)$  into  $\mathcal{E} \otimes A$ .

A  $\mathbb{G}$ -equivariant Hilbert bimodule is a Hilbert bimodule with a  $\mathbb{G}$ -coaction  $\epsilon: \mathcal{E} \rightarrow \tilde{\mathcal{M}}(\mathcal{E} \otimes A)$  that satisfies conditions (1)–(5) both for the left and the right Hilbert module structure. There is, however, some duplication here. Condition (5) is the same for the left and right Hilbert module structure, and the change between left and right exchanges conditions (3) and (4). Thus if both Hilbert module structures satisfy conditions (1)–(3) and (5), then they both satisfy (1)–(5). Hence the definition here and in [1] give the same notion of Hilbert bimodule.

The *linking algebra* associated to a Hilbert  $C$ -module  $\mathcal{E}$  is the algebra of compact operators on  $C \oplus \mathcal{E}$  with its block decomposition into  $\mathbb{K}(C, C) \cong C$ ,  $\mathbb{K}(C, \mathcal{E}) \cong \mathcal{E}$ ,  $\mathbb{K}(\mathcal{E}, C) \cong \mathcal{E}^*$  and  $\mathbb{K}(\mathcal{E}, \mathcal{E}) = \mathbb{K}(\mathcal{E})$ . A  $\mathbb{G}$ -coaction on  $\mathcal{E}$  induces a coaction  $\gamma': \mathbb{K}(C \oplus \mathcal{E}) \rightarrow \mathbb{K}(C \oplus \mathcal{E}) \otimes A$  that is compatible with this block decomposition by [1, Proposition 2.7];  $\gamma'$  restricts to  $\epsilon$  and  $\gamma$  on the blocks  $\mathcal{E}$  and  $C$  in  $\mathbb{K}(C \oplus \mathcal{E})$ . Under the assumptions in [1], this coaction need not satisfy the Podleś condition, even if  $\gamma$  does. Our additional condition (4) ensures this because it is equivalent to  $\gamma'(\mathcal{E}^*) \cdot (1 \otimes A) = \mathcal{E}^* \otimes A$ , and this implies  $\gamma'(\mathbb{K}(\mathcal{E})) \cdot (1 \otimes A) = \mathbb{K}(\mathcal{E}) \otimes A$  because  $\mathbb{K}(\mathcal{E}) = \mathcal{E} \cdot \mathcal{E}^*$ . Condition (3) and the continuity of  $\gamma$  give  $\gamma'(\mathcal{E}) \cdot (1 \otimes A) = \mathcal{E} \otimes A$  and  $\gamma'(C) \cdot (1 \otimes A) = C \otimes A$ .

**Proposition 5.10.** *Let  $\mathcal{E}$  be a  $\mathbb{G}$ -equivariant Hilbert module over  $(C, \gamma)$  and let  $\mathcal{F}$  be a  $\mathbb{G}$ -equivariant Hilbert module over  $(D, \delta)$ . Let  $C' := \mathbb{K}(C \oplus \mathcal{E})$  and  $D' := \mathbb{K}(D \oplus \mathcal{F})$  with the induced continuous coactions  $\gamma'$  and  $\delta'$ . Choose a V-Heisenberg pair  $(\alpha, \beta)$  and view  $C \boxtimes_{\mathbb{V}} D$  and  $\mathbb{K}(\mathcal{E}) \boxtimes_{\mathbb{V}} \mathbb{K}(\mathcal{F})$  as  $C^*$ -subalgebras of  $\mathcal{M}(C' \otimes D' \otimes \mathbb{K}(\mathcal{H}))$ . Then*

$$\mathcal{E} \boxtimes_{\mathbb{V}} \mathcal{F} := \iota_{C'}(\mathcal{E}) \cdot \iota_{D'}(\mathcal{F}) = \iota_{D'}(\mathcal{F}) \cdot \iota_{C'}(\mathcal{E})$$

*is a Hilbert module over  $C \boxtimes_{\mathbb{V}} D$ , where the right  $C \boxtimes_{\mathbb{V}} D$ -module structure is the multiplication in  $\mathcal{M}(C' \otimes D' \otimes \mathbb{K}(\mathcal{H}))$ , and the  $C \boxtimes_{\mathbb{V}} D$ -valued inner product is  $\langle \xi, \eta \rangle := \xi^* \cdot \eta$ . Furthermore,*

$$\mathbb{K}(\mathcal{E} \boxtimes_{\mathbb{V}} \mathcal{F}) \cong \mathbb{K}(\mathcal{E}) \boxtimes_{\mathbb{V}} \mathbb{K}(\mathcal{F}).$$

*Proof.* All this follows from Lemma 3.20. Lemma 3.20 for  $X = \mathcal{E}$ ,  $Y = \mathcal{F}$  asserts  $\iota_{C'}(\mathcal{E}) \cdot \iota_{D'}(\mathcal{F}) = \iota_{D'}(\mathcal{F}) \cdot \iota_{C'}(\mathcal{E})$ . To check that  $\mathcal{E} \boxtimes_{\mathbb{V}} \mathcal{F}$  is closed under right

multiplication under  $C \boxtimes_V D$ , we compute

$$\begin{aligned} (\mathcal{E} \boxtimes_V \mathcal{F}) \cdot (C \boxtimes_V D) &= \iota_{C'}(\mathcal{E}) \cdot \iota_{D'}(\mathcal{F}) \cdot \iota_{D'}(D) \cdot \iota_{C'}(C) \\ &= \iota_{C'}(\mathcal{E}) \cdot \iota_{D'}(\mathcal{F} \cdot D) \cdot \iota_{C'}(C) = \iota_{D'}(\mathcal{F} \cdot D) \cdot \iota_{C'}(\mathcal{E}) \cdot \iota_{C'}(C) \\ &= \iota_{D'}(\mathcal{F}) \cdot \iota_{C'}(\mathcal{E}) = \mathcal{E} \boxtimes_V \mathcal{F}. \end{aligned}$$

Similar computations give

$$\begin{aligned} (\mathcal{E} \boxtimes_V \mathcal{F})^* \cdot (\mathcal{E} \boxtimes_V \mathcal{F}) &\subseteq C \boxtimes_V D, \\ (\mathcal{E} \boxtimes_V \mathcal{F}) \cdot (\mathcal{E} \boxtimes_V \mathcal{F})^* &= \mathbb{K}(C) \boxtimes_V \mathbb{K}(D). \end{aligned}$$

The first line completes the proof that  $\mathcal{E} \boxtimes_V \mathcal{F}$  is a Hilbert module over  $C \boxtimes_V D$ . The second line says that  $\mathbb{K}(\mathcal{E} \boxtimes_V \mathcal{F}) \cong \mathbb{K}(\mathcal{E}) \boxtimes_V \mathbb{K}(\mathcal{F})$ .  $\square$

Let  $\mathcal{E}_1$  and  $\mathcal{E}_2$  be  $\mathbb{G}$ -equivariant Hilbert modules over  $C$  and let  $S: \mathcal{E}_1 \rightarrow \mathcal{E}_2$  be an adjointable operator. We want to construct an induced adjointable operator

$$S \boxtimes_V \text{id}_{\mathcal{F}}: \mathcal{E}_1 \boxtimes_V \mathcal{F} \rightarrow \mathcal{E}_2 \boxtimes_V \mathcal{F}.$$

We may view  $S$  as an adjointable operator on  $\mathcal{E} := \mathcal{E}_1 \oplus \mathcal{E}_2$  that vanishes on  $\mathcal{E}_2$  and has image contained in  $\mathcal{E}_1$ . There is a canonical unital  $*$ -homomorphism

$$\mathbb{B}(\mathcal{E}) \cong \mathcal{M}(\mathbb{K}(\mathcal{E})) \rightarrow \mathcal{M}(\mathbb{K}(\mathcal{E}) \boxtimes_V \mathbb{K}(\mathcal{F})) \cong \mathcal{M}(\mathbb{K}(\mathcal{E} \boxtimes_V \mathcal{F})) \cong \mathbb{B}(\mathcal{E} \boxtimes_V \mathcal{F}).$$

We apply it to  $S$  and then notice that the resulting operator is the extension by zero of an adjointable operator  $\mathcal{E}_1 \boxtimes_V \mathcal{F} \rightarrow \mathcal{E}_2 \boxtimes_V \mathcal{F}$ . This defines  $S \boxtimes_V \text{id}_{\mathcal{F}}$ . The map  $S \mapsto S \boxtimes_V \text{id}_{\mathcal{F}}$  is a unital, strictly continuous  $*$ -homomorphism.

A similar construction turns an adjointable operator  $T: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  between  $\mathbb{H}$ -equivariant Hilbert  $D$ -modules  $\mathcal{F}_1$  and  $\mathcal{F}_2$  into an adjointable operators

$$\text{id}_{\mathcal{E}} \boxtimes_V T: \mathcal{E} \boxtimes_V \mathcal{F}_1 \rightarrow \mathcal{E} \boxtimes_V \mathcal{F}_2.$$

The adjointable operators  $S \boxtimes_V \text{id}_{\mathcal{F}}$  and  $\text{id}_{\mathcal{E}} \boxtimes_V T$  usually do not commute with each other, so that  $S \boxtimes_V T$  is not defined unambiguously. However, if  $S$  and  $T$  are both equivariant and unitary, then they are compatible with all structure that is used to define  $\mathcal{E}_1 \boxtimes_V \mathcal{F}_1$  and  $\mathcal{E}_2 \boxtimes_V \mathcal{F}_2$  and hence must induce an isomorphism

$$S \boxtimes_V T: \mathcal{E}_1 \boxtimes_V \mathcal{F}_1 \rightarrow \mathcal{E}_2 \boxtimes_V \mathcal{F}_2.$$

Indeed, the following lemma shows that  $S \boxtimes_V \text{id}_{\mathcal{F}}$  and  $\text{id}_{\mathcal{E}} \boxtimes_V T$  commute whenever  $S$  or  $T$  is equivariant.

**Lemma 5.11.** *Let  $x \in \mathcal{M}(C)$  and  $y \in \mathcal{M}(D)$  and assume that  $x$  is  $\mathbb{G}$ -invariant or that  $y$  is  $\mathbb{H}$ -invariant, that is,  $\gamma(x) = x \otimes 1$  or  $\delta(y) = y \otimes 1$ . Then  $[\iota_C(x), \iota_D(y)] = 0$  in  $C \boxtimes_V D$ .*

*Proof.* If  $x$  is  $\mathbb{G}$ -invariant, then  $\iota_C(x) = x \otimes 1 \otimes 1$  in  $\mathcal{M}(C \otimes D \otimes \mathbb{K}(\mathcal{H}))$ . This commutes with  $\iota_D(y) \in \mathcal{M}(D \otimes \mathbb{K}(\mathcal{H}))_{23}$  because it lives in a different leg. The argument for  $\mathbb{H}$ -invariant  $y$  is the same.  $\square$

Now we turn from Hilbert modules to correspondences.

A  $\mathbb{G}$ -equivariant correspondence from  $C_1$  to  $C_2$  is a  $\mathbb{G}$ -equivariant Hilbert module  $\mathcal{E}$  over  $C_2$  with a nondegenerate representation of  $C_1$ , that is, with a morphism  $f: C_1 \rightarrow \mathbb{B}(\mathcal{E}) = \mathcal{M}(\mathbb{K}(\mathcal{E}))$ . Usually we do not mention  $f$  and instead equip  $\mathcal{E}$  with the left  $C_1$ -module structure given by  $f$ ; thus a correspondence is a bimodule with a  $C_2$ -valued right inner product and a  $\mathbb{G}$ -coaction with suitable properties.

Let  $\mathcal{E}$  with  $f: C_1 \rightarrow \mathbb{B}(\mathcal{E})$  be a  $\mathbb{G}$ -equivariant correspondence from  $C_1$  to  $C_2$  and let  $\mathcal{F}$  with  $g: D_1 \rightarrow \mathbb{B}(\mathcal{F})$  be a  $\mathbb{H}$ -equivariant correspondence from  $D_1$  to  $D_2$ . Then we get a Hilbert module  $\mathcal{E} \boxtimes_V \mathcal{F}$  over  $C_2 \boxtimes_V D_2$  and an induced morphism

$$C_1 \boxtimes_V D_1 \rightarrow \mathcal{M}(\mathbb{K}(\mathcal{E}) \boxtimes_V \mathbb{K}(\mathcal{F})) \cong \mathcal{M}(\mathbb{K}(\mathcal{E} \boxtimes_V \mathcal{F})) \cong \mathbb{B}(\mathcal{E} \boxtimes_V \mathcal{F})$$

by Proposition 5.10. This gives a correspondence from  $C_1 \boxtimes_V D_1$  to  $C_2 \boxtimes_V D_2$ .

Let  $S: \mathcal{E}_1 \rightarrow \mathcal{E}_2$  and  $T: \mathcal{F}_1 \rightarrow \mathcal{F}_2$  are isomorphisms of equivariant correspondences (that is, equivariant unitaries commuting with the left module structures). Then

$$S \boxtimes_V T: \mathcal{E}_1 \boxtimes_V \mathcal{F}_1 \rightarrow \mathcal{E}_2 \boxtimes_V \mathcal{F}_2$$

is an isomorphism of correspondences. Thus our construction descends to isomorphism classes of correspondences.

Next we consider the composition of correspondences. Let  $C_i$  for  $i = 1, 2, 3$  be  $\mathbb{G}$ - $C^*$ -algebras and let  $D_i$  for  $i = 1, 2, 3$  be  $\mathbb{H}$ - $C^*$ -algebras; let  $\mathcal{E}_1$  be a  $\mathbb{G}$ -equivariant correspondence from  $C_1$  to  $C_2$ , let  $\mathcal{E}_2$  be a  $\mathbb{G}$ -equivariant correspondence from  $C_2$  to  $C_3$ , let  $\mathcal{F}_1$  be an  $\mathbb{H}$ -equivariant correspondence from  $D_1$  to  $D_2$ , and let  $\mathcal{F}_2$  be an  $\mathbb{H}$ -equivariant correspondence from  $D_2$  to  $D_3$ . The composite correspondences  $\mathcal{E}_1 \otimes_{C_2} \mathcal{E}_2$  and  $\mathcal{F}_1 \otimes_{D_2} \mathcal{F}_2$  are again equivariant (see [1, Proposition 2.10], our extra continuity condition is easily checked).

**Lemma 5.12.** *There is a natural isomorphism of correspondences*

$$(\mathcal{E}_1 \otimes_{C_2} \mathcal{E}_2) \boxtimes_V (\mathcal{F}_1 \otimes_{D_2} \mathcal{F}_2) \cong (\mathcal{E}_1 \boxtimes_V \mathcal{F}_1) \otimes_{C_2 \boxtimes_V D_2} (\mathcal{E}_2 \boxtimes_V \mathcal{F}_2).$$

*Proof.* Let us assume that  $C_i$  is a subalgebra of  $\mathbb{B}(\mathcal{E}_i)$  for  $i = 1, 2$  (we can make these representations faithful by taking direct sums with suitable correspondences, and then argue in the end that the result remains true without these additional summands). The direct sum

$$\mathcal{E}' := C_3 \oplus \mathcal{E}_2 \oplus (\mathcal{E}_1 \otimes_{C_2} \mathcal{E}_2)$$

is a  $\mathbb{G}$ -equivariant Hilbert  $C_3$ -module on which the  $\mathbb{G}$ - $C^*$ -algebras  $C_i$  for  $i = 1, 2, 3$  and the  $\mathbb{G}$ -equivariant Hilbert modules  $\mathcal{E}_1$ ,  $\mathcal{E}_2$  and  $\mathcal{E}_1 \otimes_{C_2} \mathcal{E}_2$  act by adjointable operators. Namely,  $C_1$  acts by the given left actions on  $\mathcal{E}_1$  and  $\mathcal{E}_1 \otimes_{C_2} \mathcal{E}_2$  and by zero on  $\mathcal{E}_2$ ;  $C_2$  acts by the given left action on  $\mathcal{E}_2$  and by zero on the other summands;  $C_3$  acts on itself by left multiplication and by zero on the other summands;  $\mathcal{E}_2$  and  $\mathcal{E}_1 \otimes_{C_2} \mathcal{E}_2$  act by the isomorphisms  $\mathcal{E}_2 \cong \mathbb{K}(C_3, \mathcal{E}_2)$  and  $\mathcal{E}_1 \otimes_{C_2} \mathcal{E}_2 \cong \mathbb{K}(C_3, \mathcal{E}_1 \otimes_{C_2} \mathcal{E}_2)$  on  $C_3$  and by zero on the other summands; and  $\mathcal{E}_1$  acts on  $\mathcal{E}_2$  by the map

$$\mathcal{E}_1 \rightarrow \mathbb{B}(\mathcal{E}_2, \mathcal{E}_1 \otimes_{C_2} \mathcal{E}_2), \quad \xi \mapsto T_\xi,$$

with  $T_\xi(\eta) := \xi \otimes \eta$  for all  $\eta \in \mathcal{E}_2$ ,  $\xi \in \mathcal{E}_1$ , and  $\mathcal{E}_1$  acts by zero on the other summands.

These representations are nicely compatible in the following sense: bimodule structures on our Hilbert modules are always represented by composition of adjointable operators, and inner products are always represented by  $\langle x, y \rangle := x^* \circ y$ . Hence they extend to representations of the linking algebras  $\mathbb{K}(C_2 \oplus \mathcal{E}_1)$  of  $\mathcal{E}_1$ ,  $\mathbb{K}(C_3 \oplus \mathcal{E}_2)$  of  $\mathcal{E}_2$ , and  $\mathbb{K}(C_3 \oplus \mathcal{E}_1 \otimes_{C_2} \mathcal{E}_2)$  of  $\mathcal{E}_1 \otimes_{C_2} \mathcal{E}_2$ .

Let us assume similarly that  $D_i \subseteq \mathbb{B}(\mathcal{F}_i)$  for  $i = 1, 2$ , and let us embed the  $\mathbb{H}$ - $C^*$ -algebras  $D_i$  for  $i = 1, 2, 3$  and the  $\mathbb{H}$ -equivariant Hilbert modules  $\mathcal{F}_1$ ,  $\mathcal{F}_2$  and  $\mathcal{F}_1 \otimes_{D_2} \mathcal{F}_2$  in a similar fashion into  $\mathbb{B}(\mathcal{F}')$  with

$$\mathcal{F}' := D_3 \oplus \mathcal{F}_2 \oplus \mathcal{F}_1 \otimes_{D_2} \mathcal{F}_2.$$

The tensor products  $C_i \boxtimes_V D_i$ ,  $\mathcal{E}_i \boxtimes_V \mathcal{F}_i$  and  $(\mathcal{E}_1 \otimes_{C_2} \mathcal{E}_2) \boxtimes_V (\mathcal{F}_1 \otimes_{D_2} \mathcal{F}_2)$  are all embedded into the multiplier algebra of

$$\mathbb{K}(\mathcal{E}') \boxtimes_V \mathbb{K}(\mathcal{F}') \cong \mathbb{K}(\mathcal{E}' \boxtimes_V \mathcal{F}')$$

by Proposition 5.6 and Proposition 5.10.

The construction in Proposition 5.10 also shows that, in this representation, the bimodule structures on the Hilbert modules  $\mathcal{E}_i \boxtimes_V \mathcal{F}_i$  and  $(\mathcal{E}_1 \otimes_{C_2} \mathcal{E}_2) \boxtimes_V (\mathcal{F}_1 \otimes_{D_2} \mathcal{F}_2)$  are given by composition, and the inner products by  $\langle x, y \rangle := x^* \circ y$ . In such a situation, the composite correspondence  $(\mathcal{E}_1 \boxtimes_V \mathcal{F}_1) \otimes_{C_2 \boxtimes_V D_2} (\mathcal{E}_2 \boxtimes_V \mathcal{F}_2)$  is realised

concretely as  $(\mathcal{E}_1 \boxtimes_V \mathcal{F}_1) \cdot (\mathcal{E}_2 \boxtimes_V \mathcal{F}_2)$  with the bimodule structure over  $C_1 \boxtimes_V D_1$  and  $C_3 \boxtimes_V D_3$  given by composition and inner product  $\langle x, y \rangle := x^* \cdot y$ .

Lemma 3.20 gives

$$\begin{aligned} (\mathcal{E}_1 \otimes_{C_2} \mathcal{E}_2) \boxtimes_V (\mathcal{F}_1 \otimes_{C_2} \mathcal{F}_2) &= \iota_{\mathbb{K}(\mathcal{E}')}(\mathcal{E}_1) \cdot \iota_{\mathbb{K}(\mathcal{E}')}(\mathcal{E}_2) \cdot \iota_{\mathbb{K}(\mathcal{F}')}(\mathcal{F}_1) \cdot \iota_{\mathbb{K}(\mathcal{F}')}(\mathcal{F}_2) \\ &= \iota_{\mathbb{K}(\mathcal{E}')}(\mathcal{E}_1) \cdot \iota_{\mathbb{K}(\mathcal{F}')}(\mathcal{F}_1) \cdot \iota_{\mathbb{K}(\mathcal{E}')}(\mathcal{E}_2) \cdot \iota_{\mathbb{K}(\mathcal{F}')}(\mathcal{F}_2) = (\mathcal{E}_1 \boxtimes_V \mathcal{F}_1) \cdot (\mathcal{E}_2 \boxtimes_V \mathcal{F}_2). \end{aligned}$$

This finishes the proof.  $\square$

**5.5. Cocycle conjugacy.** We now consider the special case of equivariant Morita–Rieffel equivalence where only the  $\mathbb{G}$ -coaction is changed to an equivalent one by a cocycle:

**Definition 5.13** ([2, Definition 0.4]). A  $\gamma$ -cocycle is a unitary  $u \in \mathcal{M}(C \otimes A)$  with

$$(5.14) \quad u_{12}(\gamma \otimes \text{id}_A)u = (\text{id}_C \otimes \Delta_A)u \quad \text{in } \mathcal{U}(C \otimes A \otimes A).$$

We can only treat cocycles that satisfy an extra Podleś condition:

**Lemma 5.15.** Let  $u \in \mathcal{U}(C \otimes A)$  be a  $\gamma$ -cocycle. Define a morphism  $\gamma_u := \text{Ad}_u \circ \gamma: C \rightarrow C \otimes A$ . This is a continuous coaction of  $\mathbb{G}$  if and only if

$$(5.16) \quad \gamma(C) \cdot u^* \cdot (1_C \otimes A) = C \otimes A.$$

*Proof.* The morphism  $\gamma_u$  is faithful because  $\gamma$  is. We check that it is a comodule structure:

$$(\text{id}_C \otimes \Delta_A)(u\gamma(c)u^*) = u_{12}((\varphi \otimes \text{id}_A)(u\gamma(c)u^*)u_{12}^* = (\gamma_u \otimes \text{id}_A)\gamma_u(c)$$

for all  $c \in C$ ; the first equality uses (5.14) and (2.16) for  $\gamma$ ; the second equality again uses (5.14) for all  $c \in C$ .

Since  $u \in \mathcal{U}(C \otimes A)$  we have  $u(C \otimes A) = C \otimes A$ . Hence (5.16) is equivalent to the Podleś condition  $u\gamma(C)u^* \cdot (1 \otimes A) = C \otimes A$  for  $\gamma_u$ .  $\square$

The following result generalises [2, Proposition 7.6].

**Theorem 5.17.** Let  $u$  be a  $\gamma$ -cocycle and let  $v$  be a  $\delta$ -cocycle. Assume both satisfy the Podleś condition (5.16). Define the coactions  $\gamma_u$  and  $\delta_v$  as above. Then

$$(C, \gamma) \boxtimes_V (D, \delta) \cong (C, \gamma_u) \boxtimes_V (D, \delta_v).$$

This isomorphism is not one of crossed products, that is, it is not compatible with the embeddings of  $C$  and  $D$ .

*Proof.* Let  $\mathcal{E}$  be  $C$  viewed as a Hilbert module over itself. Define the coaction  $\epsilon: \mathcal{E} \rightarrow \tilde{\mathcal{M}}(\mathcal{E} \otimes A)$  by  $\epsilon(c) := u \cdot \gamma(c)$ . We claim that this gives a  $\mathbb{G}$ -equivariant Hilbert  $C$ -module. Conditions (1)–(3) in Definition 5.8 are immediate. Condition (4) is equivalent to (5.16) by taking adjoints, and (5) is equivalent to the cocycle condition (5.14).

Since  $\mathcal{E} = C$  as a Hilbert module, the left multiplication action of  $C$  gives an isomorphism  $C \cong \mathbb{K}(\mathcal{E})$ . The induced  $\mathbb{G}$ -coaction on  $\mathbb{K}(\mathcal{H})$  is, however, not equivalent to  $\gamma$  but to  $\gamma_u$ :  $\gamma_u(c_1) \cdot \epsilon(c_2) = \epsilon(c_1 c_2)$  for all  $c_1, c_2 \in C$ .

Similarly, let  $\mathcal{F}$  be  $D$  viewed as a Hilbert module over itself, with the  $\mathbb{H}$ -coaction  $\varphi: \mathcal{F} \rightarrow \tilde{\mathcal{M}}(\mathcal{F} \otimes B)$  defined by  $\varphi(d) := v \cdot \delta(d)$ . Then  $\mathbb{K}(\mathcal{F}) \cong D$  with induced coaction  $\delta_v$ . Now Proposition 5.10 gives

$$(C, \gamma_u) \boxtimes_V (D, \delta_v) \cong \mathbb{K}(\mathcal{E} \boxtimes_V \mathcal{F}).$$

The identity maps  $C \rightarrow \mathcal{E}$  and  $D \rightarrow \mathcal{F}$  are (non-equivariant) unitary operators. They give unitary operators

$$(C, \gamma) \boxtimes_V (D, \delta) \rightarrow (\mathcal{E}, \epsilon) \boxtimes_V (D, \delta) \rightarrow (\mathcal{E}, \epsilon) \boxtimes_V (\mathcal{F}, \varphi)$$

of Hilbert  $(C, \gamma) \boxtimes_V (D, \delta)$ -modules. Conjugating by this unitary gives a C\*-algebra isomorphism

$$(C, \gamma) \boxtimes_V (D, \delta) \cong \mathbb{K}(C \boxtimes_V D) \rightarrow \mathbb{K}(\mathcal{E} \boxtimes_V \mathcal{F}).$$

Now compose this with the isomorphism  $\mathbb{K}(\mathcal{E} \boxtimes_V \mathcal{F}) \cong (C, \gamma_u) \boxtimes_V (D, \gamma_v)$ .  $\square$

We describe the isomorphism above more explicitly. To simplify notation, we treat only  $u$  and assume  $v = 1$ . The linking algebra for  $\mathcal{E}$  is  $\mathbb{M}_2(C)$  with the  $\mathbb{G}$ -coaction

$$\begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \mapsto \begin{pmatrix} \gamma(c_{11}) & u\gamma(c_{12}) \\ \gamma(c_{21})u^* & u\gamma(c_{22})u^* \end{pmatrix}.$$

The upper left and lower right corners are  $(C, \gamma)$  and  $(C, \gamma_u)$ , respectively. Thus  $(C, \gamma) \boxtimes_V (D, \delta)$  and  $(C, \gamma_u) \boxtimes_V (D, \delta)$  are subalgebras of  $\mathbb{M}_2(C) \boxtimes D$ .

Conjugation by the partial isometry  $s = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$  and its adjoint gives isomorphisms between the two corners  $C \subseteq \mathbb{M}_2(C)$ . The strictly continuous extension of  $\iota_{\mathbb{M}_2(C)}$  maps  $s$  to a partial isometry in  $\mathbb{M}_2(C) \boxtimes_V D$ . Conjugation by this partial isometry and its adjoint restricts to isomorphisms between  $(C, \gamma) \boxtimes_V (D, \delta)$  and  $(C, \gamma_u) \boxtimes_V (D, \delta)$ .

Call a continuous coaction *inner* if it is a cocycle-twist of the trivial coaction.

**Corollary 5.18.** *The crossed product  $(C, \gamma) \boxtimes_V (D, \delta)$  is isomorphic to  $C \otimes D$  if  $\gamma$  or  $\delta$  is inner.*

*Proof.* Without loss of generality, let  $u \in \mathcal{M}(C \otimes A)$  be a cocycle for the trivial coaction  $\tau(c) := c \otimes 1$  and let  $\gamma = \tau_u$ . The cocycle  $u$  satisfies (5.16) by Lemma 5.15. Now Theorem 5.17 and Example 3.24 give  $(C, \gamma) \boxtimes_V D \cong (C, \tau) \boxtimes_V D \cong C \otimes D$ .  $\square$

*Example 5.19.* Let  $U^{\mathcal{H}}$  and  $U^{\mathcal{K}}$  be corepresentations of  $A$  and  $B$  on Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$ . These are cocycles for the trivial action on  $\mathbb{K}(\mathcal{H})$ . Assume (5.16) to get continuous coactions on  $\mathbb{K}(\mathcal{H})$  and  $\mathbb{K}(\mathcal{K})$ . Then

$$\mathbb{K}(\mathcal{H}) \boxtimes_V \mathbb{K}(\mathcal{K}) \cong \mathbb{K}(\mathcal{H}) \otimes \mathbb{K}(\mathcal{K}) \cong \mathbb{K}(\mathcal{H} \otimes \mathcal{K}).$$

This explains the Hilbert space realisation of  $C \boxtimes_V D$  in Theorem 4.3 in the case where the corepresentations  $U^{\mathcal{H}}$  and  $U^{\mathcal{K}}$  used there satisfy the technical condition (5.16). Then we get a faithful morphism  $C \boxtimes_V D \rightarrow \mathbb{K}(\mathcal{H}) \boxtimes_V \mathbb{K}(\mathcal{K})$  from Proposition 5.6. When we identify  $\mathbb{K}(\mathcal{H}) \boxtimes_V \mathbb{K}(\mathcal{K}) \cong \mathbb{K}(\mathcal{H} \otimes \mathcal{K})$  as above, we get a faithful representation of  $C \boxtimes_V D$  on  $\mathcal{H} \otimes \mathcal{K}$ .

## 6. EXAMPLES OF TWISTED TENSOR PRODUCTS

We show in Section 6.1 that the skew-commutative tensor product of  $\mathbb{Z}/2$ -graded C\*-algebras is a special case of our theory.

In Section 6.2 we consider the case where both  $A$  and  $B$  are duals of locally compact groups; in particular, this covers the case where  $A$  and  $B$  are locally compact Abelian groups. Here we understand bicharacters in a classical way, and we show that  $C \boxtimes_V D$  for any bicharacter is a Rieffel deformation of  $C \otimes D$ .

In Section 6.3, we treat crossed products for coactions and construct the dual coaction on a crossed product using the functoriality of  $\boxtimes_V$ .

**6.1. Skew-commutative tensor products.** Let  $\mathbb{Z}/2 = \{0, 1\}$  be the two-element group. Let  $\mathbb{G} = \mathbb{H}$  be  $C^*(\mathbb{Z}/2)$  with the usual comultiplication. Thus a  $\mathbb{G}$ -coaction on a C\*-algebra  $C$  is a  $\mathbb{Z}/2$ -grading: a decomposition  $C = C_0 \oplus C_1$  into involutive, closed, linear subspaces  $C_0$  and  $C_1$  of even and odd elements such that

$$C_i \cdot C_j = C_{i+j \bmod 2}, \quad C_i^* = C_i.$$

Equivalently,  $\alpha'(c_0 + c_1) := c_0 - c_1$  for  $c_i \in C_i$  defines an involutive \*-automorphism of  $C$ .

The *skew-commutative tensor product* of two  $\mathbb{Z}/2$ -graded  $C^*$ -algebras  $C$  and  $D$  is defined in [4, §2.6] by imposing the commutation relation that  $c \in C$  and  $d \in D$  anti-commute if both are odd, and commute if one of them is even. This leads to the  $*$ -algebra structure

$$\begin{aligned} (c_1 \hat{\odot} d_1) \cdot (c_2 \hat{\odot} d_2) &:= (-1)^{\deg(c_2) \cdot \deg(d_1)} c_1 c_2 \hat{\odot} d_1 d_2, \\ (c \hat{\odot} d)^* &:= (-1)^{\deg(c) \cdot \deg(d)} c^* \hat{\odot} d^* \end{aligned}$$

on the algebraic tensor product  $C \hat{\odot} D$  of  $C$  and  $D$ . The skew-commutative  $C^*$ -tensor product  $C \hat{\otimes} D$  is the completion of the  $*$ -algebra  $C \hat{\odot} D$  in the  $C^*$ -norm

$$(6.1) \quad \|x\| := \sup \frac{(\rho \hat{\otimes} \lambda)(y^* \cdot x^* \cdot x \cdot y)}{(\rho \hat{\otimes} \lambda)(y^* \cdot y)}$$

over all non-zero elements  $y \in C \hat{\odot} D$  and all graded states  $\rho \in C^*$  and  $\lambda \in D^*$  (graded means that  $\rho$  and  $\lambda$  vanish on  $C_1$  and  $D_1$ , respectively); here the products and adjoints are with respect to the  $*$ -algebra structure on  $C \hat{\odot} D$ .

The obvious formulas define morphisms  $\iota_C: C \rightarrow C \hat{\otimes} D$  and  $\iota_D: D \rightarrow C \hat{\otimes} D$ , so that  $C \hat{\otimes} D$  is a crossed product of  $C$  and  $D$ . We want to show that  $C \hat{\otimes} D \cong C \boxtimes_V D$  for a suitable bicharacter  $V \in \mathcal{U}(\hat{A} \otimes \hat{A})$ .

The dual  $\hat{\mathbb{G}}$  is the group  $\mathbb{Z}/2$ , so that  $\hat{A} \otimes \hat{B} \cong C(\mathbb{Z}/2 \times \mathbb{Z}/2)$  and a bicharacter  $V$  is a bicharacter  $\mathbb{Z}/2 \times \mathbb{Z}/2 \rightarrow \mathbb{T}$  in a more classical sense. The unique non-trivial bicharacter is defined by  $V(1, 1) = -1$  and  $V(i, j) = 1$  if  $i = 0$  or  $j = 0$ .

**Theorem 6.2.** *Let  $C$  and  $D$  be  $\mathbb{Z}/2$ -graded  $C^*$ -algebras and let  $V$  be the non-trivial bicharacter in  $C(\mathbb{Z}/2 \times \mathbb{Z}/2)$ . Then the crossed product  $(C \boxtimes_V D, \iota_C, \iota_D)$  of  $C$  and  $D$  is naturally isomorphic to their skew-commutative tensor product.*

*Proof.* A covariant representation of  $C$  is given by a  $\mathbb{Z}/2$ -graded Hilbert space  $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$  and a representation  $\varphi: C \rightarrow \mathbb{B}(\mathcal{H})$  with  $\varphi(c_i)(\mathcal{H}_j) \subseteq \mathcal{H}_{i+j}$  for all  $i, j \in \mathbb{Z}/2$ . We choose such a faithful covariant representation of  $A$  and a faithful covariant representation  $\psi: D \rightarrow \mathbb{B}(\mathcal{K})$  on a  $\mathbb{Z}/2$ -graded Hilbert space  $\mathcal{K} = \mathcal{K}_0 \oplus \mathcal{K}_1$ .

Since  $\hat{A}^u = \hat{A}$ , the unitary  $Z$  that is used in the Hilbert space description of  $C \boxtimes_V D$  is described most easily by (4.6). This gives  $Z(\xi \otimes \eta) = -\xi \otimes \eta$  if  $\xi \in \mathcal{H}_1$  and  $\eta \in \mathcal{K}_1$ , and  $Z(\xi \otimes \eta) = \xi \otimes \eta$  if  $\xi \in \mathcal{H}_0$  or  $\eta \in \mathcal{K}_0$ . Thus  $\Sigma Z: \mathcal{H} \otimes \mathcal{K} \rightarrow \mathcal{K} \otimes \mathcal{H}$  is the braiding operator from the Koszul sign rule. The representations  $\varphi_1$  and  $\tilde{\psi}_2$  in Theorem 4.3 are

$$\varphi_1(c)(\xi \otimes \eta) = (\varphi(c)\xi) \otimes \eta, \quad \tilde{\psi}_2(d)(\xi \otimes \eta) = (-1)^{\deg(d) \deg(\xi)} \xi \otimes \psi(d)\eta,$$

as expected from the Koszul sign rule. It remains to show that this pair of representations of  $C$  and  $D$  yields a faithful representation of the skew-commutative tensor product  $C \hat{\otimes} D$ . It is clear that we get a  $*$ -representation of  $C \hat{\odot} D$ .

We must show that, for any  $x \in C \hat{\odot} D$ , its operator norm on  $\mathcal{H} \otimes \mathcal{K}$  is equal to the norm defined in (6.1). The GNS-representation for a graded state  $\rho: C \rightarrow \mathbb{C}$  on the Hilbert space  $L^2(C, \rho)$  is a covariant representation if we let  $L^2(C, \rho)_i$  be the closure of  $C_i$  in  $L^2(C, \rho)$ . The direct sum of these GNS-representations for all graded states is a faithful representation of  $C$  because any state on  $C_0$  extends to a state on  $C$  and a representation of  $C$  is faithful once it is faithful on  $C_0$ . Since  $C \boxtimes_V D$  does not depend on the covariant representations, we may assume that  $\varphi$  and  $\psi$  are these direct sums of covariant GNS-representations of  $C$  and  $D$ , respectively. The resulting representations  $\varphi_1$  and  $\tilde{\psi}_2$  are block diagonal with respect to the direct sum over the graded states  $\rho$  and  $\lambda$ , and each block is obtained from the GNS-representation for the pair of graded states  $\rho$  and  $\lambda$ . The elements  $y \in C \hat{\odot} D$  in (6.1) form a dense subset of the Hilbert space  $L^2(C, \rho) \otimes L^2(D, \lambda)$ , and the expression in (6.1) for fixed  $\rho$  and  $\lambda$  is precisely the norm quotient  $\|x \cdot y\|/\|y\|$ ,



where  $x \cdot y$  is defined using  $\varphi_1 \odot \tilde{\psi}_2$ . Hence the norm in (6.1) is exactly the operator norm for a particular choice of the covariant representations  $\varphi$  and  $\psi$ .  $\square$

**6.2. General group coactions.** Now we consider the case where  $A = C_r^*(G)$  and  $B = C_r^*(H)$  for two locally compact groups  $G$  and  $H$  with the usual comultiplications. Thus coactions of  $\mathbb{G}$  and  $\mathbb{H}$  are coactions of these groups  $G$  and  $H$  in the usual sense. We are going to identify  $C \boxtimes_V D$  with a Rieffel deformation of the commutative tensor product  $C \otimes D$  in the sense of [5]. To begin with, we reduce to the case where both  $G$  and  $H$  are Abelian.

The dual quantum groups are the groups  $G$  and  $H$ , respectively. Since  $\hat{A} \otimes \hat{B} = C_0(G \times H)$ , a bicharacter  $V \in \mathcal{U}(\hat{A} \otimes \hat{B})$  is a bicharacter  $V: G \times H \rightarrow \mathbb{T}$  in the classical sense. Since  $\mathbb{T}$  is commutative,  $V(g, h)$  vanishes if  $g$  or  $h$  is a commutator. Hence  $V$  descends to a continuous biadditive map  $V': G^{\text{ab}} \times H^{\text{ab}} \rightarrow \mathbb{T}$  on the Abelianisations  $G^{\text{ab}}$  and  $H^{\text{ab}}$ , giving us a bicharacter  $V^{\text{ab}} \in \mathcal{U}(C_0(G^{\text{ab}} \times H^{\text{ab}}))$ . The quotient maps  $G \rightarrow G^{\text{ab}}$  and  $H \rightarrow H^{\text{ab}}$  are quantum group morphisms. They allow us to turn the given coactions of  $G$  and  $H$  on  $C$  and  $D$  into coactions of  $G^{\text{ab}}$  and  $H^{\text{ab}}$ , respectively. Theorem 5.2 shows  $C \boxtimes_V D = C \boxtimes_{V^{\text{ab}}} D$ , where the right hand side uses only the induced coactions of  $G^{\text{ab}}$  and  $H^{\text{ab}}$ . Hence we may without loss of generality assume that  $G$  and  $H$  are Abelian locally compact groups.

Let  $\hat{G}$  and  $\hat{H}$  be their Pontryagin duals. We may also view a bicharacter as a continuous group homomorphism  $G \rightarrow \hat{H}$  or  $H \rightarrow \hat{G}$  by fixing one of the two variables. This makes it easy to list all bicharacters for two given Abelian locally compact groups  $G$  and  $H$ . Coactions of  $G$  and  $H$  are equivalent to actions of  $\hat{G}$  and  $\hat{H}$ , respectively. Thus  $C$  and  $D$  carry actions of  $\hat{G}$  and  $\hat{H}$ , respectively. The commutative tensor product

$$E := C \otimes D$$

inherits an action of  $\Gamma := \hat{G} \times \hat{H}$ . The bicharacter  $V: G \times H \rightarrow \mathbb{T}$  yields a bicharacter

$$\Psi: \hat{\Gamma} \times \hat{\Gamma} \rightarrow \mathbb{T}, \quad \Psi((g_1, h_1), (g_2, h_2)) := V(g_2, h_1)^{-1}.$$

Any bicharacter is also a two-cocycle, which may be used as a deformation parameter for Rieffel deformations. Here we define Rieffel deformations following Kasprzak [5] using crossed products and Landstad theory.

**Theorem 6.3.**  *$C \boxtimes_V D$  is naturally isomorphic to the Rieffel deformation of  $E$  with respect to  $\Psi$ .*

*Proof.* Pick faithful representations of  $C \rtimes \hat{G}$  and  $D \rtimes \hat{H}$  on Hilbert spaces  $\mathcal{H}$  and  $\mathcal{K}$ , respectively. These give faithful covariant representations of  $C$  and  $D$ , which we use to represent  $C \boxtimes_V D$  faithfully on  $\mathcal{H} \otimes \mathcal{K}$ . They also generate a faithful representation of

$$E \rtimes \Gamma \cong (C \rtimes \hat{G}) \otimes (D \rtimes \hat{H})$$

on  $\mathcal{H} \otimes \mathcal{K}$ . The description of the operator  $Z$  in (4.6) shows that  $Z \in \mathcal{M}(E \rtimes \Gamma)$ . Thus  $C \boxtimes_V D$  is contained in  $\mathcal{M}(E \rtimes \Gamma)$ , generated by the canonical embeddings of  $C$  and  $D$ , with the latter twisted by  $\text{Ad}_Z$ .

The Rieffel deformation  $E^\Psi$  of  $E$  with respect to  $\Psi$  is described in [5] as a subalgebra of  $\mathcal{M}(E \rtimes \Gamma)$  as well. We will use formal properties of  $E^\Psi$  to deduce that  $E^\Psi = C \boxtimes_V D$  as  $C^*$ -subalgebras of  $E \rtimes \Gamma$ .

Since  $E = (C \otimes 1) \cdot (1 \otimes D)$ , [6, Lemma 3.4] yields

$$(6.4) \quad E^\Psi = (C \otimes 1)^\Psi \cdot (1 \otimes D)^\Psi;$$

Here we let  $\Gamma$  act on  $C$  and  $D$  by letting  $H$  act trivially on  $C$  and  $G$  trivially on  $D$ .

The deformation procedure in [5] uses the unitaries

$$U_{g,h} \in C(G \times H, \mathbb{T}), \quad U_{g,h}(g_1, h_1) := \Psi((g_1, h_1), (g, h)) = V(g, h_1)^{-1}$$

for  $(g, h), (g_1, h_1) \in G \times H = \hat{\Gamma}$ . Since  $\hat{H}$  acts trivially on  $C$ , we have  $C \rtimes \Gamma \cong (C \rtimes \hat{G}) \otimes C^*(\hat{H})$ . Hence the unitaries  $U_{g,h}$  are mapped to central elements in  $C \rtimes \Gamma$ . In this case, the Rieffel deformation does nothing, that is,  $C^\Psi = C$  as subalgebras of  $\mathcal{M}(C \rtimes \Gamma)$ . Thus  $(C \otimes 1)^\Psi$  in (6.4) is represented on  $\mathcal{H} \otimes \mathcal{K}$  by  $\varphi_1(C)$ .

Now define another two-cocycle on  $\hat{\Gamma}$  by  $\Psi'((g_1, h_1), (g_2, h_2)) := V(g_1, h_2)$ ; the Rieffel deformation for  $\Psi'$  involves the unitaries  $U'_{g,h}(g_1, h_1) := \Psi'((g_1, h_1), (g, h)) = V(g_1, h)$ , which are mapped to central elements in  $D \rtimes \Gamma \cong C^*(\hat{G}) \otimes (D \rtimes \hat{H})$ . Therefore,  $D^{\Psi'} = D$  as subalgebras of  $\mathcal{M}(D \rtimes \Gamma)$ .

The two-cocycles  $\Psi$  and  $\Psi'$  are cohomologous: let  $f(g, h) = V(g, h)^{-1}$ , then

$$\partial f((g_1, h_1), (g_2, h_2)) := \frac{f(g_1 g_2, h_1 h_2)}{f(g_1, h_1) f(g_2, h_2)} = V(g_2, h_1)^{-1} V(g_1, h_2)^{-1};$$

thus  $\partial f \cdot \Psi' = \Psi$ . Now [5, Lemmas 3.4 and 3.5] yield

$$D^\Psi = (D^{\Psi'})^{\partial f} = D^{\partial f} = f D f^*$$

as  $C^*$ -subalgebras of  $D \rtimes \Gamma$ . Here  $f$  is viewed as a unitary element of  $C^*(\Gamma) \subseteq D \rtimes \Gamma$ .

Equation (4.6) shows that the representation  $D \rtimes \Gamma \rightarrow E \rtimes \Gamma \rightarrow \mathbb{B}(\mathcal{H} \otimes \mathcal{K})$  maps  $f$  to  $Z$ . Thus  $(1 \otimes D)^\Psi = 1 \otimes f D f^*$  in (6.4) is represented on  $\mathcal{H} \otimes \mathcal{K}$  by  $\tilde{\psi}_2(D)$ . Finally, (6.4) becomes  $E^\Psi = \varphi_1(C) \cdot \tilde{\psi}_2(D) \cong C \boxtimes_{\mathbb{V}} D$  as desired.  $\square$

**6.3. Crossed products.** Consider the special case where  $\mathbb{H} = \hat{\mathbb{G}}$ ,  $V = W^A \in \mathcal{U}(\hat{A} \otimes A)$ ,  $D = \hat{A}$ ,  $\delta = \hat{\Delta}_A: \hat{A} \rightarrow \hat{A} \otimes \hat{A}$ . We claim that  $(C, \gamma) \boxtimes_{W^A} (\hat{A}, \hat{\Delta}_A)$  is the *reduced crossed product* of  $(C, \gamma)$ . More precisely, the reduced crossed product  $C \rtimes_r \hat{A}$  comes equipped with canonical morphisms  $\iota_C: C \rightarrow C \rtimes_r \hat{A}$  and  $\iota_{\hat{A}}: \hat{A} \rightarrow C \rtimes_r \hat{A}$ , such that  $(C \rtimes_r \hat{A}, \iota_C, \iota_{\hat{A}})$  is a crossed product in the sense of Definition 2.1. We claim that this is equivalent to  $(C, \gamma) \boxtimes_{W^A} (\hat{A}, \hat{\Delta}_A)$  as a crossed product.

Let  $(\pi, \hat{\pi})$  be a  $\mathbb{G}$ -Heisenberg pair on the Hilbert space  $\mathcal{H}$  of the special form in Example 3.9; that is,  $W_{\hat{\pi}\pi} = \mathbb{W}$  is a multiplicative unitary generating  $\mathbb{G}$ .

**Theorem 6.5.** *There is a faithful morphism  $\rho: (C, \gamma) \boxtimes_{W^A} (\hat{A}, \hat{\Delta}_A) \rightarrow C \otimes \mathbb{K}(\mathcal{H})$  with  $\rho \circ \iota_C = \gamma_{1\pi}$  and  $\rho \circ \iota_{\hat{A}} = \hat{\pi}_2$ , where*

$$\begin{aligned} \gamma_{1\pi}: C &\rightarrow \mathcal{M}(C \otimes \mathbb{K}(\mathcal{H})), & c &\mapsto (\text{id}_C \otimes \pi)\gamma(c), \\ \hat{\pi}_2: \hat{A} &\rightarrow \mathcal{M}(C \otimes \mathbb{K}(\mathcal{H})), & \hat{a} &\mapsto 1_C \otimes \hat{\pi}(\hat{a}). \end{aligned}$$

Reduced crossed products for locally compact quantum groups (with Haar weights) are defined using the pair of representations  $\gamma_{1\pi}$  and  $\hat{\pi}_2$  above, where  $\pi$  and  $\hat{\pi}$  are the regular representations. Thus Theorem 6.5 provides an isomorphism  $(C, \gamma) \boxtimes_{W^A} (\hat{A}, \hat{\Delta}_A) \rightarrow C \rtimes_r \hat{A}$  for locally compact quantum groups. For general  $C^*$ -quantum groups, we may now *define* the reduced crossed product using the pair of representations  $\gamma_{1\pi}$  and  $\hat{\pi}_2$  and get the isomorphism  $(C, \gamma) \boxtimes_{W^A} (\hat{A}, \hat{\Delta}_A) \rightarrow C \rtimes_r \hat{A}$  in complete generality.

*Proof of Theorem 6.5.* Since

$$\gamma_{1\pi} \otimes \text{id}_{\mathbb{K}(\mathcal{H})}: C \otimes \mathbb{K}(\mathcal{H}) \rightarrow C \otimes \mathbb{K}(\mathcal{H}) \otimes \mathbb{K}(\mathcal{H})$$

is a faithful morphism, the pair of representations  $(\gamma_{1\pi}, \hat{\pi}_2)$  generates a faithful representation of  $C \boxtimes \hat{A}$  if and only the pair  $((\gamma_{1\pi} \otimes \text{id}_{\mathbb{K}(\mathcal{H})}) \circ \gamma_{1\pi}, (\gamma_{1\pi} \otimes \text{id}_{\mathbb{K}(\mathcal{H})}) \circ \hat{\pi}_2)$  does so. We have  $(\gamma_{1\pi} \otimes \text{id}_{\mathbb{K}(\mathcal{H})}) \circ \hat{\pi}_2(\hat{a}) = \hat{\pi}_3(\hat{a})$  and

$$(\gamma_{1\pi} \otimes \text{id}_{\mathbb{K}(\mathcal{H})}) \gamma_{1\pi}(c) = (\gamma \otimes \text{id}_A) \gamma(c)_{1\pi\pi} = (\text{id}_C \otimes \Delta_A) \gamma(c)_{1\pi\pi} = \mathbb{W}(\text{id} \otimes \pi) \gamma(c)_{12} \mathbb{W}^*.$$

Let  $\Sigma_{23}$  be the coordinate flip. Conjugating both representations by the same unitary  $\Sigma_{23} \mathbb{W}^*$  gives a unitarily equivalent pair of representations. Hence we may

further replace  $\gamma_{1\pi}$  and  $\hat{\pi}_2$  by the representations  $c \mapsto (\text{id} \otimes \pi)\gamma(c)_{13}$  of  $C$  and

$$\hat{a} \mapsto \Sigma_{23} \mathbb{W}^* \hat{\pi}(\hat{a})_3 \mathbb{W} \Sigma_{23} = \hat{\Delta}(\hat{a})_{\hat{\pi}\hat{\pi}}$$

of  $\hat{A}$ ; here we use the standard description of  $\hat{\Delta}$  in terms of  $\mathbb{W}$ .

Thus we arrive at the pair of representations  $(\text{id}_C \otimes \hat{\pi} \otimes \text{id}_{\mathbb{K}(\mathcal{H})})\iota_C$  and  $(\text{id}_C \otimes \hat{\pi} \otimes \text{id}_{\mathbb{K}(\mathcal{H})})\iota_{\hat{A}}$  with  $\iota_C = \gamma_{1\pi}$  and  $\iota_{\hat{A}} = (\hat{\Delta}_A)_{2\hat{\pi}}$  in  $C \otimes \hat{A} \otimes \mathbb{K}(\mathcal{H})$ . Since  $\hat{\pi}$  is faithful, this pair is equivalent to  $(\iota_C, \iota_{\hat{A}})$ . Since this pair defines the crossed product  $C \boxtimes \hat{A}$ , we see that  $(\gamma_{1\pi}, \hat{\pi}_2)$  generates an equivalent crossed product as claimed.  $\square$

Viewing the reduced crossed product as a special case of  $\boxtimes$  gives us more freedom because we may also tensor  $(C, \gamma)$  with other  $\hat{\mathbb{G}}$ -C\*-algebras and use functoriality. We now describe the dual coaction in this way, using the functoriality of  $\boxtimes$ .

The map  $\hat{\Delta}: \hat{A} \rightarrow \hat{A} \otimes \hat{A}$  is  $\hat{\mathbb{G}}$ -equivariant if  $\hat{\mathbb{G}}$  coacts on  $\hat{A} \otimes \hat{A}$  by  $\text{id} \otimes \hat{\Delta}: \hat{A} \otimes \hat{A} \rightarrow \hat{A} \otimes \hat{A} \otimes \hat{A}$ . By the functoriality of  $\boxtimes$ , this equivariant morphism induces a morphism

$$\hat{\delta}: C \rtimes \hat{A} \cong C \boxtimes \hat{A} \rightarrow C \boxtimes (\hat{A} \otimes \hat{A}) \cong \hat{A} \otimes (C \boxtimes \hat{A}) \cong \hat{A} \otimes (C \rtimes \hat{A});$$

here we use Lemma 3.25 in the second variable to pull out the first factor  $\hat{A}$ .

**Lemma 6.6.** *The map  $\hat{\delta}: C \rtimes \hat{A} \rightarrow \hat{A} \otimes (C \rtimes \hat{A})$  is a continuous left  $\hat{\mathbb{G}}$ -coaction.*

*Proof.* The comodule property of  $\hat{\delta}$  follows from the coassociativity of  $\hat{\Delta}$  and the functoriality of  $\boxtimes$ . The map  $\hat{\delta}$  is faithful by Proposition 5.6. The Podleś condition for  $\hat{\delta}$  follows because  $(\hat{A} \otimes 1)\hat{\Delta}(\hat{A}) = \hat{A} \otimes \hat{A}$ : apply  $\iota_{\hat{A} \otimes \hat{A}}$  to this equality.  $\square$

This coaction is uniquely determined by the conditions  $\hat{\delta}(\iota_C(c)) = 1 \otimes \iota_C(c)$  and  $\hat{\delta}(\iota_{\hat{A}}(\hat{a})) = (\text{id}_{\hat{A}} \otimes \iota_{\hat{A}})\hat{\Delta}$ . The same conditions characterise the *dual coaction*. Thus we have indeed constructed the dual coaction.

The functoriality of  $\boxtimes$  in the first variable gives us the usual functoriality of reduced crossed products.

General tensor products  $C \boxtimes_{\mathbb{W}} (D, \delta)$  are closely related to the crossed product through Lemma 2.18, which shows that  $\delta: D \rightarrow D \otimes \hat{A}$  is a  $\hat{\mathbb{G}}$ -equivariant embedding for the coaction  $\text{id}_D \otimes \hat{\Delta}$  on  $D \otimes \hat{A}$ . By Proposition 5.6 and Lemma 3.25, this induces a faithful morphism

$$C \boxtimes_{\mathbb{W}} D \rightarrow C \boxtimes_{\mathbb{W}} (D \otimes \hat{A}) \cong D \otimes (C \boxtimes_{\mathbb{W}} \hat{A}) \cong D \otimes (C \rtimes_{\mathbb{r}} \hat{A}).$$

Now we consider once again the general situation of two quantum groups  $\mathbb{G} = (A, \Delta_A)$  and  $\mathbb{H} = (B, \Delta_B)$  and a bicharacter  $V \in \mathcal{U}(\hat{A} \otimes \hat{B})$ .

**Theorem 6.7.** *View  $\hat{A} \otimes \hat{B}$  as a subalgebra of  $(C \rtimes_{\mathbb{r}} \hat{A}) \otimes (D \rtimes_{\mathbb{r}} \hat{B})$  via  $\iota_{\hat{A}} \otimes \iota_{\hat{B}}$ . The embeddings*

$$\begin{aligned} (\iota_C)_1: C &\rightarrow (C \rtimes_{\mathbb{r}} \hat{A}) \otimes (D \rtimes_{\mathbb{r}} \hat{B}), & c &\mapsto \iota_C(c) \otimes 1, \\ \text{Ad}_{V^*} \circ (\iota_D)_2: D &\rightarrow (C \rtimes_{\mathbb{r}} \hat{A}) \otimes (D \rtimes_{\mathbb{r}} \hat{B}), & d &\mapsto V^*(1 \otimes \iota_D(d))V, \end{aligned}$$

*induce a faithful morphism*

$$C \boxtimes_V D \rightarrow (C \rtimes_{\mathbb{r}} \hat{A}) \otimes (D \rtimes_{\mathbb{r}} \hat{B}).$$

*Proof.* Choose faithful representation  $\varphi_0: C \rightarrow \mathbb{B}(\mathcal{H}_0)$ . Let  $(\pi, \hat{\pi})$  be a  $\mathbb{G}$ -Heisenberg pair as in Example 3.9, acting on a Hilbert space  $\mathcal{H}_\pi$ . Let  $\mathcal{H} := \mathcal{H}_0 \otimes \mathcal{H}_\pi$ . Then we get a faithful representation  $\varphi_0 \otimes \text{id}$  of  $C \otimes \mathbb{K}(\mathcal{H}_\pi)$  on  $\mathcal{H}$ . This restricts to a faithful representation of  $\varphi': C \rtimes_{\mathbb{r}} \hat{A} \rightarrow \mathbb{B}(\mathcal{H})$  because the latter is the crossed product generated by the morphisms  $\gamma_{1\pi}: C \rightarrow C \otimes \mathbb{K}(\mathcal{H}_\pi)$  and  $\hat{\pi}_2: \hat{A} \rightarrow C \otimes \mathbb{K}(\mathcal{H}_\pi)$ .

Now we compare the construction of a covariant representation of  $(C, \gamma)$  in Example 4.5. We see that this covariant representation consists of  $\rho \circ \iota_C: C \rightarrow \mathbb{B}(\mathcal{H})$  and  $W_{\rho, \hat{A}}^A \in \mathcal{U}(\mathbb{K}(\mathcal{H}_\pi) \otimes A)$ . Furthermore, the representation of  $\hat{A}^u$  used later in

the proof of Theorem 4.1 is  $\rho_{\hat{A}} \circ \Lambda$  for the reducing morphism  $\Lambda: \hat{A}^u \rightarrow \hat{A}$ . (Actually, any representation of  $C \rtimes_r \hat{A}$  gives a covariant representation of  $(C, \gamma)$  in a similar way.)

Now do the same things for  $(D, \delta)$ : let  $\psi_0: D \rightarrow \mathbb{B}(\mathcal{K}_0)$  be a faithful representation; choose a  $\mathbb{H}$ -Heisenberg pair  $(\rho, \hat{\rho})$  as in Example 3.9, acting on a Hilbert space  $\mathcal{K}_\rho$ ; let  $\mathcal{K} := \mathcal{K}_0 \otimes \mathcal{K}_\rho$ ; let  $\psi'$  be the resulting faithful representation of  $D \rtimes_r \hat{B}$  on  $\mathcal{K}$ ; construct a covariant representation of  $(D, \delta)$  on  $\mathcal{K}$  as in Example 4.5.

Now Theorem 4.3 gives a faithful representation of  $C \boxtimes_V D$  on  $\mathcal{H} \otimes \mathcal{K}$ , generated by the representations  $\varphi_1$  and  $\text{Ad}_Z \psi_2$ . By construction, we also get a faithful representation  $\varphi' \otimes \psi'$  of  $(C \rtimes_r \hat{A}) \otimes (D \rtimes_r \hat{B})$  on  $\mathcal{H} \otimes \mathcal{K}$ . The description of  $Z$  in (4.6) yields  $Z = (\varphi' \otimes \psi')(V^*)$ . Hence the representations  $\varphi_1$  and  $\text{Ad}_Z \psi_2$  both factor through the embedding  $\varphi' \otimes \psi'$  and the maps  $(\iota_C)_1$  and  $\text{Ad}_{V^*} \circ (\iota_D)_2$  in the statement of the theorem. We thus get a faithful morphism  $C \boxtimes_V D \rightarrow (C \rtimes_r \hat{A}) \otimes (D \rtimes_r \hat{B})$  restricting to  $(\iota_C)_1$  and  $\text{Ad}_{V^*} \circ (\iota_D)_2$  on  $C$  and  $D$ .  $\square$

For instance, in the situation of Section 6.1, this realises the skew-commutative tensor product  $C \otimes D$  as a subalgebra of  $(C \rtimes \mathbb{Z}/2) \otimes (D \rtimes \mathbb{Z}/2)$ .

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